Oracle-based approximation algorithms for the discrete resource sharing scheduling problem

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13 May, University of Tennessee

1. *PTAS* design techniques
   - Some classical techniques
   - The oracle formalism

2. The DRSSP problem

3. Oracle approximation
   - Guessing the correct oracle answer
   - Second guess : convenient subset
   - Guess approximation

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The main techniques...

Some of the main *PTAS* design techniques [SW00]:

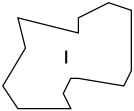
- structuring the input
- structuring the output ("extending partial small size solutions")
- structuring the execution of an algorithm ("trimmed algorithm")

Structuring the input

Given in instance I, the main ("polynomial") steps are:

- **simplify**: turn I into a more primitive instance I' . This simplification depends on the desired precision ϵ
- **solve**: determine an optimal solution Opt' for I' (in polynomial time)
- **translate back**: translate the solution Opt' for I' into an approximate solution S for I

Figure from [SW00]



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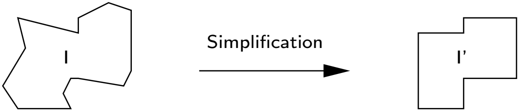

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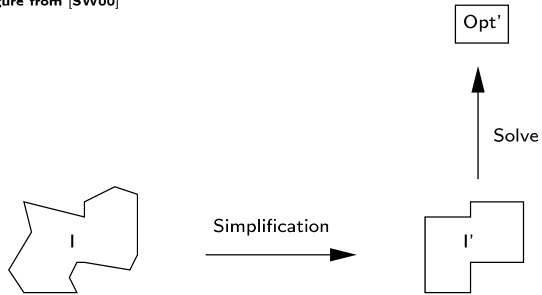

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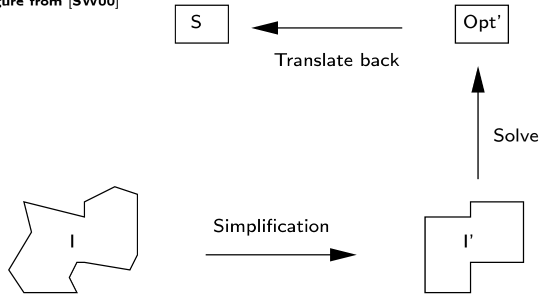

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Given in instance I, the main ("polynomial") steps are:

- **partition**: partition the feasible solution space F into a (polynomial) number of districts $F^{(1)}$, .., $F^{(d)}$. This partition depends on the desired precision ϵ.
- **find representative**: For each district $F^{(l)}$, determine a good representative $S^{(l)}$ "close" to $Opt^{(l)}$
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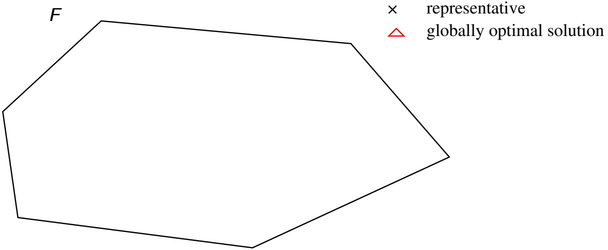

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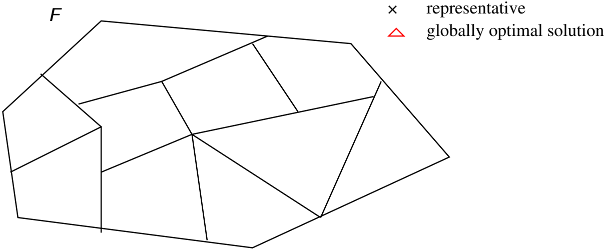

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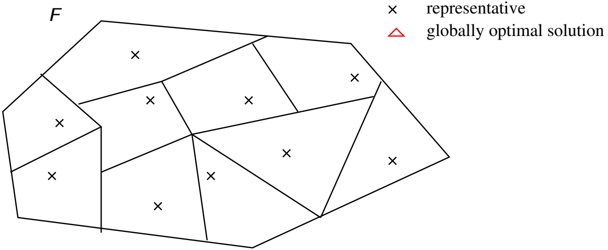

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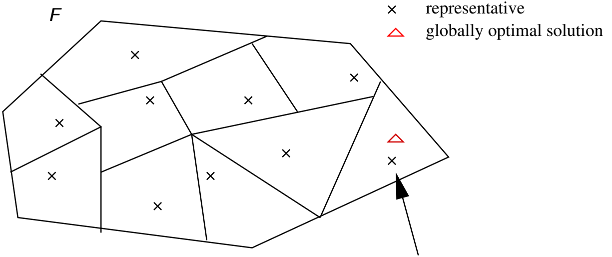

Structuring the execution of an algorithm

Given in instance I, perform a polynomial number of "meta" steps. At each step:

- **extend**: extend every partial solution of the current set
- **collapse**: according to a previously defined "grid"', collapse all the partial solutions which are in the same "box"

take the best: After the last step, we get solutions for the original problem. Select the best of all these solutions.

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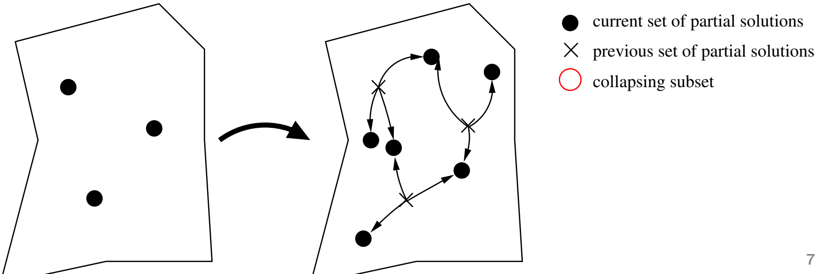

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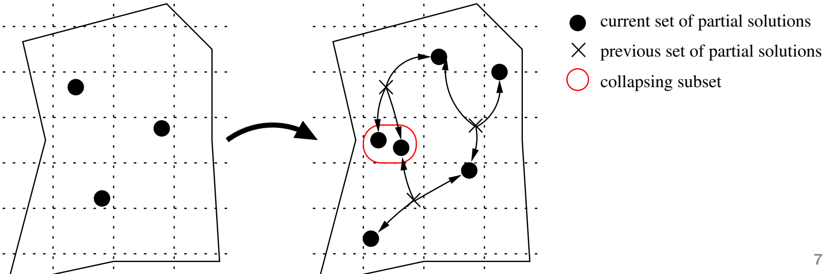

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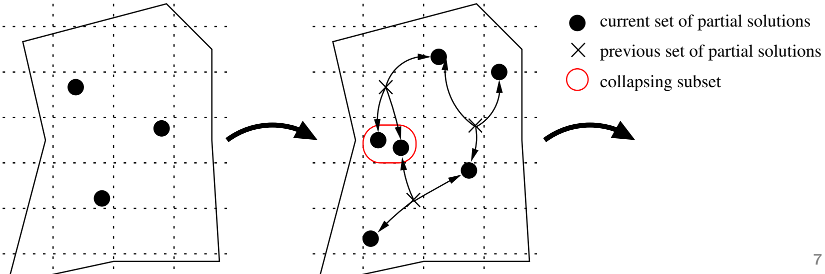

Oracle based approach

This formalism is directly inspired from complexity theory. Given in instance I, the main ("polynomial") steps are:

- **choose the question**: choose an "interesting" property P
- ask a question $Q(I)$ to the (reliable) oracle
- the oracle provides an answer $r^* \in R$ (s t. $P(Q(I), r^*)$ is true)
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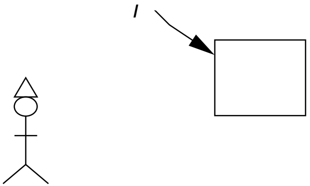
- **without the oracle**: try all the possible answers and select the best of all the $S(r), r \in R$



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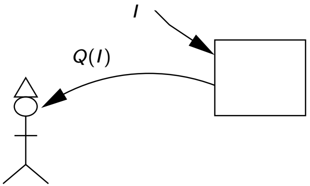

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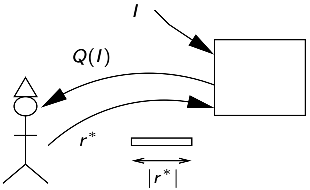

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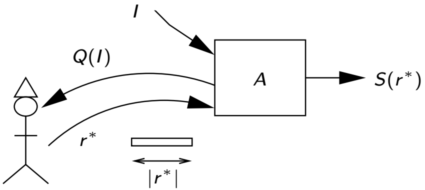

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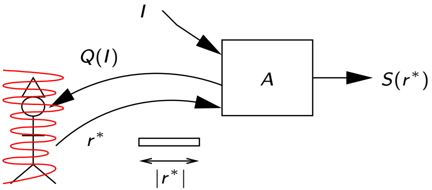

Oracle based approach

Thus, the obtained algorithm (without oracle):

- is a ρ approximation
- has a computational complexity in $O(t_A * 2^{|r^*|})$

Generally, we can choose the size $|r^*|$ (leading to different ρ), leading to approximation schemes.

The answer size is crucial !

Beside efficient (compact) representation, we will look into lossy compression.

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Oracle approach Vs structuring the output

When asking a particular type of "questions", the oracle formalism can be equivalent to the output structuring technique. An example for $P||C_{max}$:

- question : where do you schedule the biggest task (in an optimal solution)?
- provide a solution for all the possible oracle answer $r \iff$ provide a solution for every district
- the oracle answer r^* indicates a district containing an optimal solution

However, these are very special cases.

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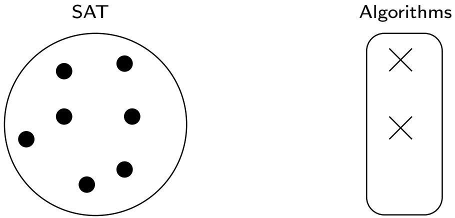
- set of algorithms
- goal: minimize the time needed to solve all the instances from the benchmark
- more than selection: combination of algorithms



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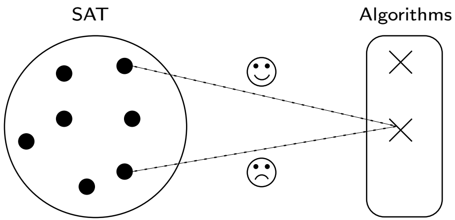

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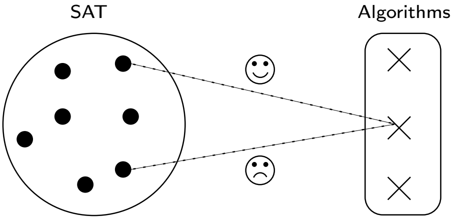

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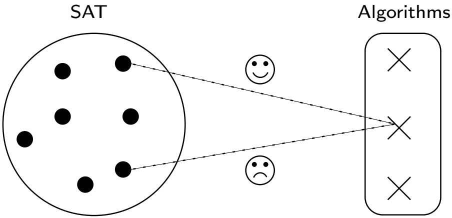

Introduction

What me mean by combination:

- one instance may be treated by several algorithms in parallel
- when a solution of an instance is found, everyone is aware
- but, the solution for an instance cannot be merged from partial solutions provided by different algorithms

Algorithm are parallel.
Parallel task model : moldable.

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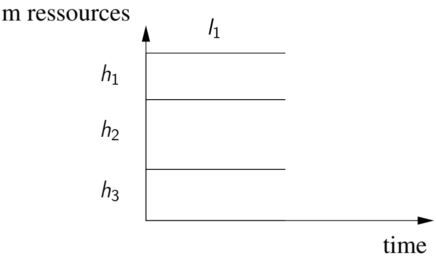

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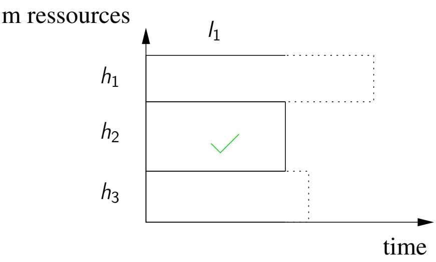

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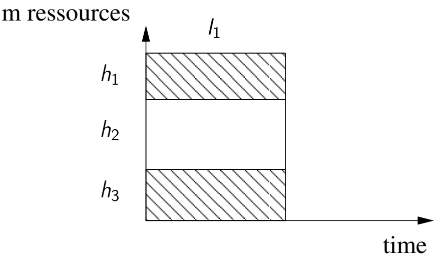

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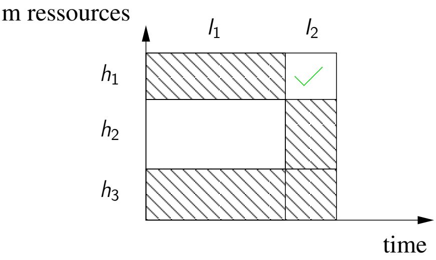

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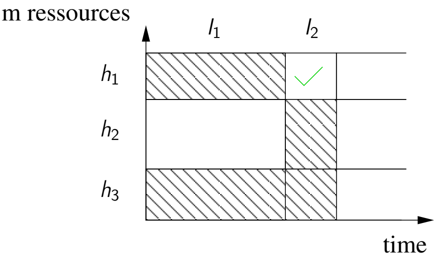

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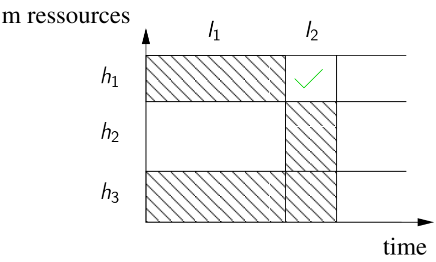

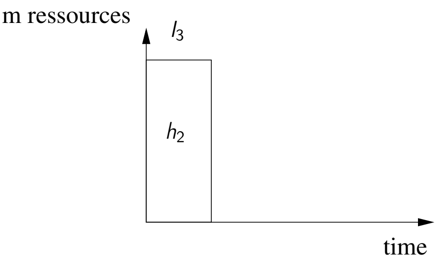

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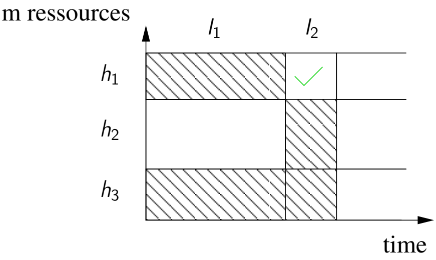

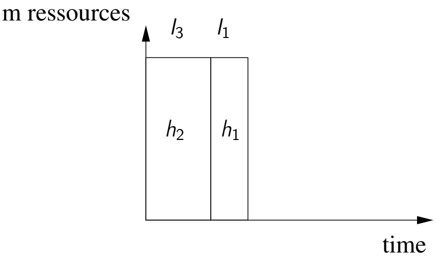

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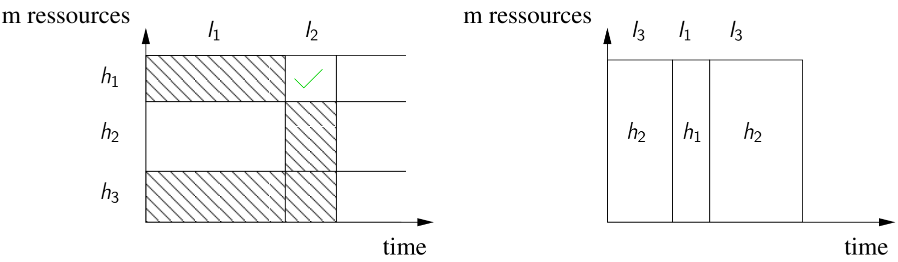

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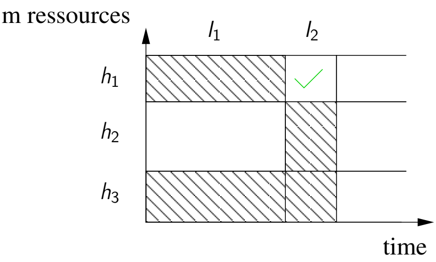

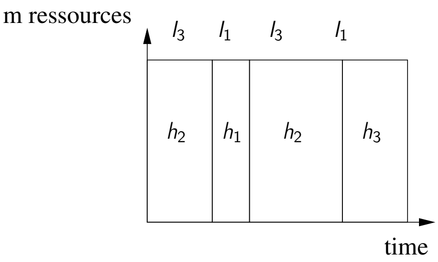

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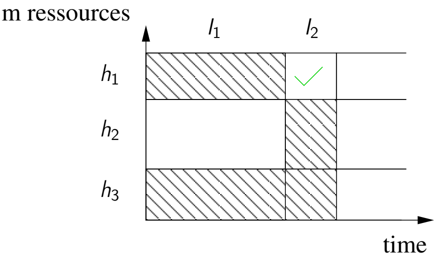
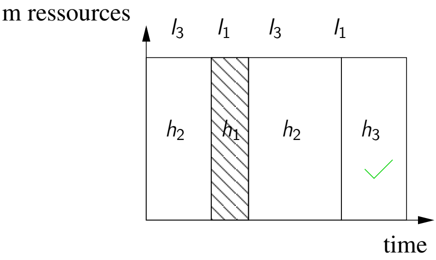




Introduction

Context :

- hybridation, algorithm portfolios
- two of the existing techniques : time sharing Vs space sharing

Space sharing assumptions (for a fixed problem P):

- a portfolio of algorithm for P is given
- there exists a finite set I of *representative* input of P
- the time needed by every algorithm to solve every instance of I is known a priori !
- the goal is to minimize the mean execution time for an instance of I

Definition of the dRSSP

Input of the discrete Resource Sharing Scheduling Problem:

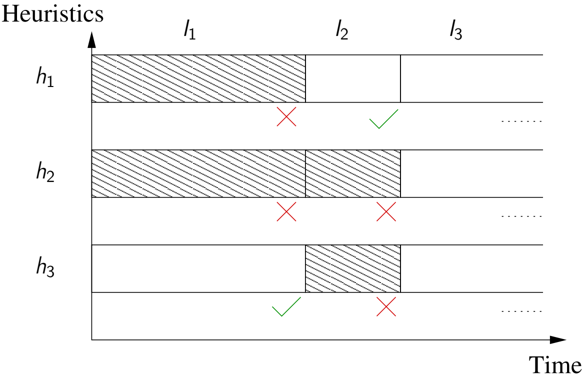
- a finite set of instances $I = \{I_1, \ldots, I_n\}$
- a finite set of heuristics $H = \{h_1, \ldots, h_k\}$
- m identical resources
- a cost $C(h_i, I_j, p) \in R^+$ for each $I_j \in I$, $h_i \in H$ and $p \in \{1, \ldots, m\}$

Continuous version ($p \in R^+$) in [SFM06].

Definition of the dRSSP

Output : an allocation $S = (S_1, \ldots, S_k)$ such that:

- $S_i \in \{0, \ldots, m\}$
- $0 < \sum_{i=1}^{k} S_i \leq m$
- S minimizes $\sum_{j=1}^{n} \min_{1 \leq i \leq k} \{C(h_i, I_j, S_i) | S_i > 0\}$



A restricted version

We study a restricted version in which :

- the cost function is linear in the number of resources $C(h_i, I_j, S_i) = \frac{C(h_i, I_j)}{S_i}$
- each heuristic must use at least one processor ($S_i \geq 1$), (well chosen portfolio)

Remark : with only the first constraint, the problem is inaproximable within a constant factor (if $m < k$).

A simple greedy algorithm

We consider the mean-allocation (MA) algorithm which simply allocates $\lfloor \frac{m}{k} \rfloor$ resources to each heuristic.

Proposition

MA is a k approximation.

Notations (given a solution S):

- let $\sigma(j) = i_0 / \frac{C(h_{i_0}, I_j)}{S_{i_0}} = \min_{1 \leq i \leq k} \frac{C(h_i, I_j)}{S_i}$ be the index of the used heuristic for instance $j \in \{1, .., n\}$ in S
- let $T(I_j) = \frac{C(h_{\sigma(j)}, I_j)}{S_{\sigma(j)}}$ be the processing time of instance j in S

A simple greedy algorithm

Proof: Let $(a, b) \in \mathbb{N}^2$ such that $m = ak + b, b < k, a \geq 1$.
$\forall j \in \{1, .., n\}$:

$$\begin{aligned} T(I_j) \leq \frac{C(h_{\sigma*(j)}, I_j)}{S_{\sigma^*(j)}} &= \frac{S^*_{\sigma^*(j)}}{S_{\sigma^*(j)}} T^*(I_j) \\ &\leq \frac{m-(k-1)}{S_{\sigma^*(j)}} T^*(I_j) \\ &= \frac{ak+b-(k-1)}{a} T^*(I_j) \leq kT^*(I_j) \end{aligned}$$

Guess 1

As a first step, we ask the correct allotment for g heuristics.

Definition

Let $G_1 = (S_1^*, \ldots, S_g^*)$, for a fixed subset of g heuristics and a fixed optimal solution S^*.

Notice that $|G_1| = glog(m)$.

We need some notations :

- let $k' = k - g$ be the number of remaining heuristics
- let $s = \Sigma_{i=1}^{g} S_i^*$ the number of processors used in the guess
- let $m' = m - s$ the number of remaining processors
- let $(a', b') \in \mathbb{N}^2$ such that $m' = a'k' + b', b' < k'$

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Algorithm MA^G

We consider the following MA^G algorithm (given any guess $G = (X_1, \ldots, X_g), X_i \geq 1$):

- allocate X_i processors to heuristic $h_i, i \in \{1, \ldots, g\}$
- applies MA on the k' others heuristics with the m' remaining processors

We will use this algorithm with $G = G_1$.

Analysis of MA^{G_1}

Proposition

MA^{G_1} is a $k - g$ approximation.

Proof:

- MA^{G_1} produces a valid solution because $a' \geq 1$
- for any instance j treated by a guessed heuristic in the optimal solution considered MA^{G_1} is even better than the optimal
- for the others, the analysis is the same as for the algorithm MA, and leads to the desired ratio

Algorithm MA_R^G

The ratio for instances treated by the guessed heuristics is unnecessarily good.
Thus, we consider mean-allocation-reassign (MA_R^G) algorithm (given any guess $G = (X_1, \ldots, X_g), X_i \geq 1$):

- allocates $X_i - \lfloor \frac{X_i}{\alpha} \rfloor$ processors to heuristic $h_i, i \in \{1, \ldots, g\}$
- applies MA on the k' others heuristics with the $m' + \Sigma_{i=1}^{g} \lfloor \frac{X_i}{\alpha} \rfloor$ remaining processors

Remark

- MA_R^G doesn't respect G
- maybe we asked the wrong question ?

Another analysis of MA

For any heuristic $h_i, i \in \{1, .., k\}$, let $T^*(h_i) = \Sigma_{j/\sigma^*(j)=i} T^*(I_j)$ be the "useful" computation time of heuristic i in the solution S^*.

$$
\begin{aligned}
T_{MA} &= \sum_{i=1}^{k} \sum_{j/\sigma^*(j)=i} T(I_j) \\
&\leq \sum_{i=1}^{k} \frac{S_i^*}{S_i} \sum_{j/\sigma^*(j)=i} T^*(I_j) \\
&= \sum_{i=1}^{k} \frac{S_i^*}{S_i} T^*(h_i) \\
&\leq Max_i(T^*(h_i)) \frac{m}{\lfloor \frac{m}{k} \rfloor} \\
&\leq Max_i(T^*(h_i))(2k-1)
\end{aligned}
$$

Guess 2

Definition

Let $G_2 = (S_1^*, \ldots, S_g^*)$, such that $T^*(h_1) \geq .. \geq T^*(h_g) \geq T^*(h_i), \forall i \in \{g+1, .., k\}$ in a fixed optimal solution S^*.

Notice that $|G_2| = glog(k) + glog(m)$.
We will use the algorithm MA^G with $G = G_2$.

Analysis of MA^{G_2}

Proposition

MA^{G_2} is a $\frac{k-1}{g}$ approximation.

Proof: We proceed as in the new analysis of *MA*:

$$
\begin{aligned}
T_{algo} &= \sum_{i=1}^{g} \sum_{j/\sigma^*(j)=i} T(I_j) + \sum_{i=g+1}^{k} \sum_{j/\sigma^*(j)=i} T(I_j) \\
&\leq \sum_{i=1}^{g} T^*(h_i) + \sum_{i=g+1}^{k} \frac{S_i^*}{S_i} T^*(h_i) \\
&= \sum_{i=1}^{k} T^*(h_i) + \sum_{i=g+1}^{k} (\frac{S_i^*}{S_i} - 1) T^*(h_i) \\
&= Opt + \underbrace{T^*(h_g)}_{\leq \frac{Opt}{g}} (\frac{m'}{a'} - k')
\end{aligned}
$$

Introduction

Goal: we want $\bar{G}$ smaller than G, without degrading too much the solution.

To solve these problems, we want:

- $\bar{S}_i \leq S_i^*$
- $\bar{S}_i = S_i^*$ for the "small" values of S_i^*

Thus, given a guess $G = (S_1^*, .., S_g^*)$:

- we choose a size j_1 bits for the significant, $j_1 \in \{1, .., \lceil \log(m) \rceil \}$
- we write $S_i^* = t_i 2^{x_i} + r_i$, with t_i encoded on j_1 bits, and $0 \leq x_i \leq \lceil \log(m) \rceil - j_1$, et $r_i \leq 2^{x_i} - 1$
- we define $\bar{S}_i = t_i 2^{x_i}$

We consider that the oracle gives $\bar{G}_2$. Notice that $|\bar{G}_2| = \Sigma_{i=1}^{g}(|t_i| + |x_i|) \leq g(j_1 + log(log(m))$.

Analysis of $MA^{\bar{G}_2}$

Proposition

$MA^{\bar{G}_2}$ is a $\beta + \frac{k-g-1}{g}$ approximation, with $1 + \frac{1}{2^{j_1-1}} = \beta$.

Proof:

- if $S_i^* \leq 2^{j_1} - 1$, then $\bar{S}_i = S_i^*$
- else, $\frac{S_i^*}{\bar{S}_i} = \frac{t_i 2^{x_i} + r_i}{t_i 2^{x_i}} \leq 1 + \frac{1}{t_i} \leq 1 + \frac{1}{2^{j_1-1}} = \beta$

Then, using the same analysis as MA^{G_2}:

$$\begin{aligned} T_{algo} &\leq \sum_{i=1}^{g} \beta T^*(h_i) + \sum_{i=g+1}^{k} \frac{S_i^*}{S_i} T^*(h_i) \\ &= \beta Opt + \underbrace{T^*(h_g)}_{\leq \frac{Opt}{g}}\left(\frac{m'}{a'} - k'\right) \end{aligned}$$

Summary

Outline of the derived approximation schemes:

algorithm	approximation ratio	complexity
MA^{G_1}	$(k-g)$	$O(m^g * kn)$
MA^{G_2}	$\frac{k-1}{g}$	$O((km)^g * kn)$
$MA^{\bar{G}_2}$	$\beta + \frac{k-g-1}{g}$	$O(k(2^{j_1} log(m))^g * kn)$

In [SFM06], k is fixed.

Conclusion

- Spatial heuristics combination
- Complexity vs. Approximation trade-off
- Partial oracle (To err is human !)

What's next ?

- real application experiments (SAT solvers)
- extend the method to other problems
- explore connections with PCP theory

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