Overview of Scheduling 1/2

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Evolution of parallel machines

An ever-increasing demand of computing power

Parallelism is an attempt to answer

Parallel algorithm design and scheduling were already difficult tasks with homogeneous machines

On heterogeneous platforms, it gets worse

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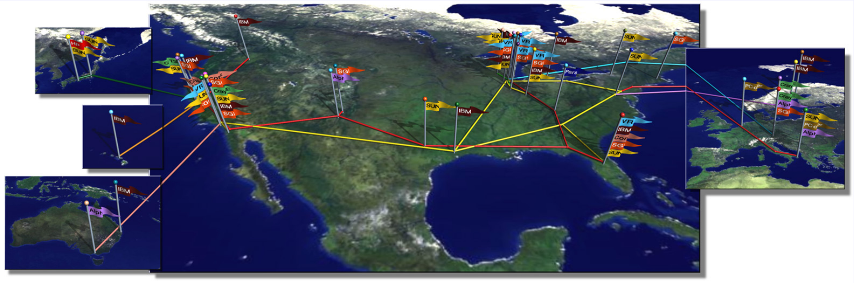

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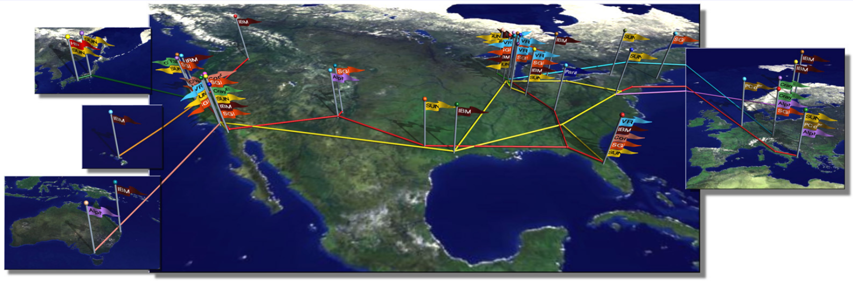

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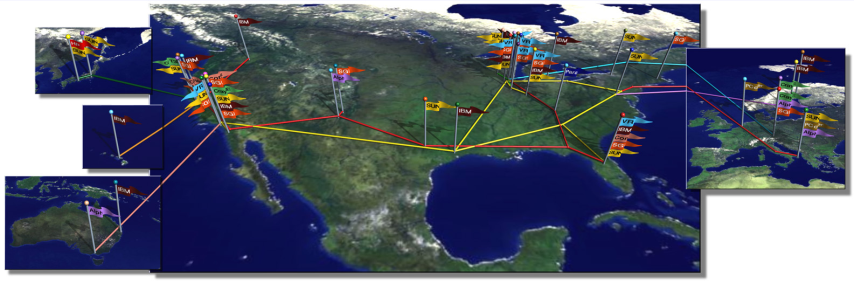

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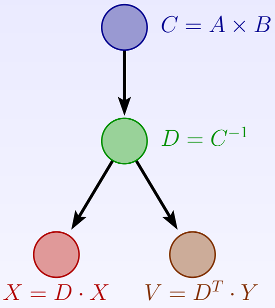
Outline

1. Background on traditional scheduling
2. Divisible Load Scheduling (or changing the task model)
3. Simulation for Grid Computing (next week)
4. Steady-State Scheduling (next week)

Outline

Traditional scheduling – Framework

- Application = DAG $G = (\mathcal{T}, E, w)$
  - $\mathcal{T}$ = set of tasks
  - E = dependence constraints
  - $w(T)$ = computational cost of task T (execution time)
  - $c(T, T')$ = communication cost (data sent from T to T')



- Platform
  - Set of p identical processors

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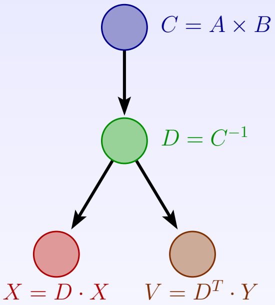

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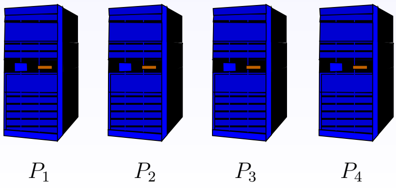
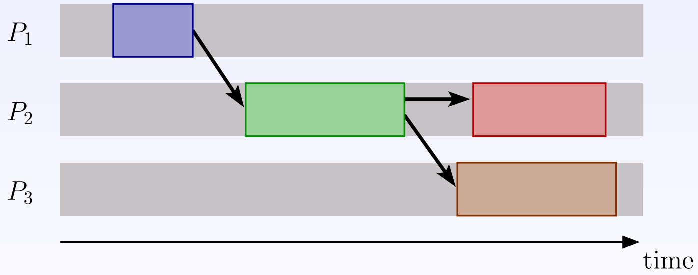


Traditional scheduling – Framework

- Schedule
  - $\sigma(T)$ = date to begin execution of task T
  - alloc(T) = processor assigned to it



Traditional scheduling – Constraints

- Data dependences If $(T, T') \in E$ then
  - if alloc(T) = alloc(T') then
    $$\sigma(T) + w(T) \leq \sigma(T')$$
  - if alloc(T) $\neq$ alloc(T') then
    $$\sigma(T) + w(T) + c(T, T') \leq \sigma(T')$$
- Resource constraints
  $$\text{alloc}(T) = \text{alloc}(T') \Rightarrow \\ [\sigma(T), \sigma(T) + w(T)[\bigcap [\sigma(T'), \sigma(T') + w(T')[= \emptyset$$

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Traditional scheduling – Objective functions

- Makespan or total execution time

$$MS(\sigma) = \max_{T \in \mathcal{T}} \left(\sigma(T) + w(T)\right)$$

- Other classical objectives:
  - Sum of completion times
  - With arrival times: maximum flow (response time), or sum flow
  - More fairness with maximum stretch, or sum stretch

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Traditional scheduling – About the model

- Simple but OK for computational resources
  - No CPU sharing, even in models with preemption
  - At most one task running per processor at any time-step
- Very crude for network resources
  - Unbounded bandwidth on any link
  - Unlimited number of simultaneous sends/receives per processor
  - Fully connected interconnection graph (clique)

In fact, model assumes infinite network capacity

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Makespan minimization

- NP-hardness
  - $Pb(p)$ NP-complete for independent tasks and no communications ($E = \emptyset$, $p = 2$ and $c = \overline{0}$)
  - $Pb(p)$ NP-complete for UET-UCT graphs ($w = c = \overline{1}$)
- Approximation algorithms
  - Without communications, list scheduling is a $(2 - \frac{1}{p})$-approximation
  - With communications, result extends to coarse-grain graphs
  - With communications, no λ-approximation in general

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List scheduling – Without communications (1/2)

- *Initialization:*
  1. Compute priority level of all tasks
  2. Priority queue = list of free tasks (tasks without predecessors) sorted by priority
  3. t is the current time step: $t = 0$.
- *While there remain tasks to execute*:
  1. Add new free tasks, if any, to the queue. If the execution of a task terminates at time step t, suppress this task from the predecessor list of all its successors. Add those tasks whose predecessor list has become empty.
  2. If there are q available processors and r tasks in the queue, remove first $\min(q, r)$ tasks from the queue and execute them; if T is one of these tasks, let $\sigma(T) = t$.
  3. Increment t.

List scheduling – Without communications (2/2)

- Priority level
  - Use critical path: longest path from the task to an exit node
  - Computed recursively by a bottom-up traversal of the graph
- Implementation details
  - Cannot iterate from $t = 0$ to $t = MS(\sigma)$ (exponential in problem size)
  - Use a heap for free tasks valued by priority level
  - Use a heap for processors valued by termination time
  - Complexity $O(|V| \log |V| + |E|)$

List scheduling – With communications (1/2)

- Priority level
  - Use pessimistic critical path: include all edge costs in the weight
  - Computed recursively by a bottom-up traversal of the graph
- MCP Modified Critical Path
  - Assign free task with highest priority to best processor
  - Best processor = finishes execution first, given already taken scheduling decisions
  - Free tasks may not be ready for execution (communication delays)
  - May explore inserting the task in empty slots of schedule
  - Complexity $O(|V| \log |V| + (|E| + |V|)p)$

List scheduling – With communications (2/2)

- ETF Earliest Finish Time
  - Dynamically recompute priorities of free tasks
  - Select free task that finishes execution first (on best processor), given already taken scheduling decisions
  - Higher complexity $O(|V|^3 p)$
  - May miss "urgent" tasks on critical path
- Other approaches
  - Two-step: clustering + load balancing
    - DSC Dominant Sequence Clustering $O((|V| + |E|) \log |V|)$
    - LLB List-based Load Balancing $O(C \log C + |V|)$ (C number of clusters generated by DSC)
  - Low-cost: FCP Fast Critical Path
    - Maintain constant-size sorted list of free tasks:
    - Best processor = first idle or the one sending last message
    - Low complexity $O(|V| \log p + |E|)$

Extending the model to heterogeneous clusters

- Task graph with n tasks $T_1, \ldots, T_n$.
- Platform with p heterogeneous processors $P_1, \ldots, P_p$.
- Computation costs:
  - w_{iq} = execution time of T_i on P_q
  - $\overline{w_i} = \frac{\sum_{q=1}^{p} w_{iq}}{p}$ average execution time of T_i
  - particular case: consistent tasks $w_{iq} = w_i \times \gamma_q$
- Communication costs:
  - data(i, j): data volume for edge $e_{ij} : T_i \rightarrow T_j$
  - v_{qr}: communication time for unit-size message from P_q to P_r (zero if $q = r$)
  - com(i, j, q, r) = data$(i, j) \times v_{qr}$ communication time from T_i executed on P_q to P_j executed on P_r
  - $\overline{\text{com}_{ij}}$ = data$(i, j) \times \frac{\sum_{1 \leq q, r \leq p, q \neq r} v_{qr}}{p(p-1)}$ average communication cost for edge $e_{ij} : T_i \rightarrow T_j$

Rewriting constraints

Dependences For $e_{ij} : T_i \to T_j$, $q = \mathsf{alloc}(T_i)$ and $r = \mathsf{alloc}(T_j)$:

$$\sigma(T_i) + w_{iq} + \mathsf{com}(i, j, q, r) \leq \sigma(T_j)$$

Resources If $q = \mathsf{alloc}(T_i) = \mathsf{alloc}(T_j)$, then

$$(\sigma(T_i) + w_{iq} \leq \sigma(T_j)) \text{ or } (\sigma(T_j) + w_{jq} \leq \sigma(T_i))$$

Makespan

$$\max_{1 \leq i \leq n} \left(\sigma(T_i) + w_{i,\mathsf{alloc}(T_i)}\right)$$

HEFT: Heterogeneous Earliest Finishing Time

1. Priority level:
   - $\text{rank}(T_i) = \overline{w_i} + \max_{T_j \in \text{Succ}(T_i)}(\overline{\text{com}_{ij}} + \text{rank}(T_j))$, where $\text{Succ}(T)$ is the set of successors of T
   - Recursive computation by bottom-up traversal of the graph
2. Allocation
   - For current task T_i, determine best processor P_q: minimize $\sigma(T_i) + w_{iq}$
   - Enforce constraints related to communication costs
   - Insertion scheduling: look for $t = \sigma(T_i)$ s.t. P_q is available during interval $[t, t + w_{iq}[$
3. Complexity: similar to MCP

New platforms, new problems, new solutions

Target platforms: Large-scale heterogeneous platforms
(networks of workstations, clusters, collections of clusters, grids, ...)

New problems

- Heterogeneity of processors (CPU power, memory)
- Heterogeneity of communication links
- Irregularity of interconnection network
- Non-dedicated platforms

Need to adapt algorithms and scheduling strategies: new objective functions, new models

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New approaches

1. Divisible load model Assume work can be arbitrarily divided into sub-tasks
2. Steady-state throughput Rather than optimizing execution from the very beginning to the very end, optimize execution core instead

Both approaches are relaxation methods: simplifying model allows to tackle more complex problems

Bibliography

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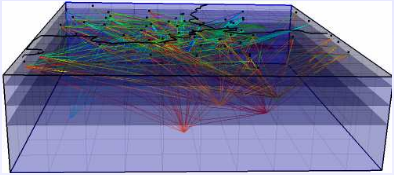
2. Divisible Load Scheduling (or changing the task model)
   - Bus network – Classical approach
   - Bus network – Divisible Load Approach
   - Star network

3. Simulation for Grid Computing (next week)

4. Steady-State Scheduling (next week)

Earth seismic tomography

- Modeling the internal structure of earth



- Validation by comparing expected propagation time of a wave in a model against collected experimental data
- Set of seismic events during 1999: 817101 events
- Original code written for a parallel machine:

```
if (rank = ROOT)
   raydata ← read n lines from data file;
MPI_Scatter(raydata, n/P, ..., rbuff, ...,
            ROOT, MPI_COMM_WORLD);
compute_work(rbuff);
```

Divisible load applications

- Divisible load applications can be divided into any number of independent pieces
- Perfectly parallel job: any sub-task can itself be processed in parallel, and on any number of workers
- Model = good approximation for applications that consist of very (very) large numbers of identical, low-granularity computations
- Large spectrum of scientific problems

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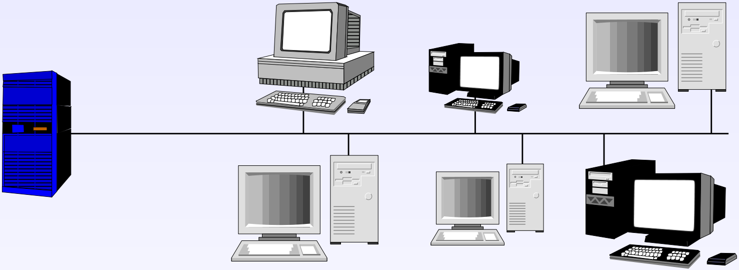
Divisible Load Scheduling (DLS)

- Applications composed of a very (very) large number of low-granularity computations
- Computation time proportional to data volume processed: linear cost model
- Independent computations: neither synchronizations nor communications

Bus network



- Identical links between master and slaves
- Slaves have different computing power

Notations

- Set P_1, ..., P_p of processors
- P_1 is the master, initially holds all data
- Total amount of work: W_{total}
- Processor P_i receives $n_i \in \mathbb{N}$ load units, where $\sum_i n_i = W_{\text{total}}$
  Time for one load unit on P_i: w_i
  Execution time on P_i: $n_i w_i$
- Communication time of one load-unit from P_1 to P_i: c
  One-port model: P_1 serially sends one message to each slave

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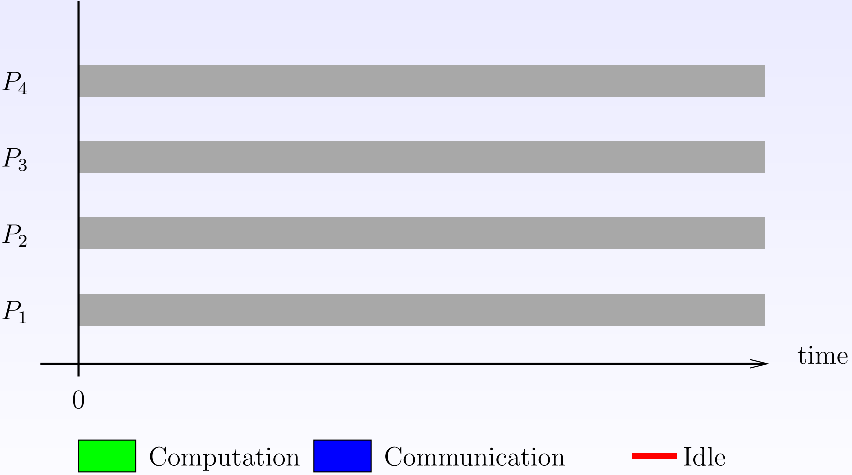
Assessing the model

- One-port hypothesis is realistic ☺
- Linear cost model is simple ☹ but acceptable ☺ for
  - large problems
  - one-round scenarios

Scheduling example



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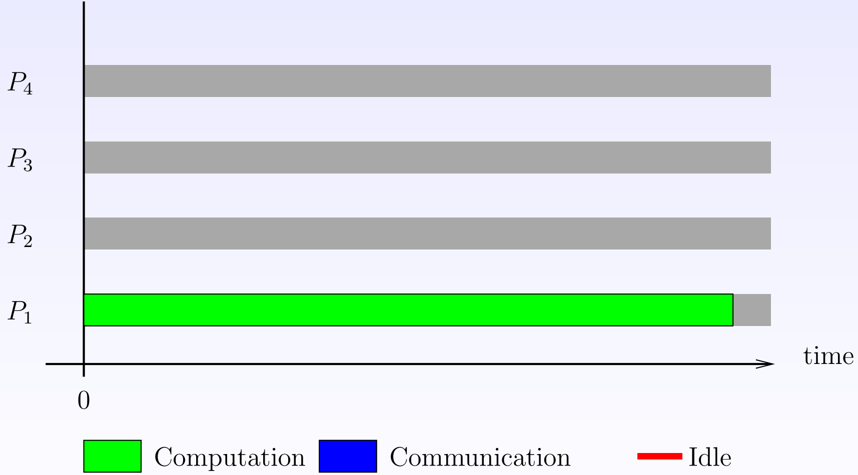

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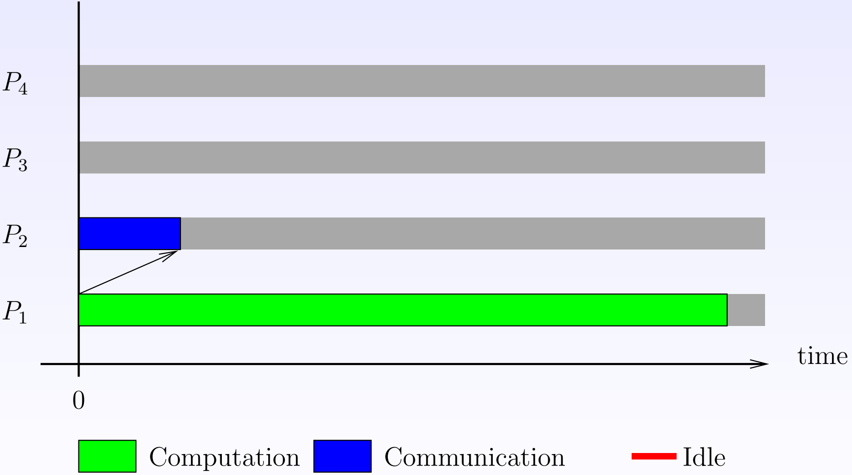

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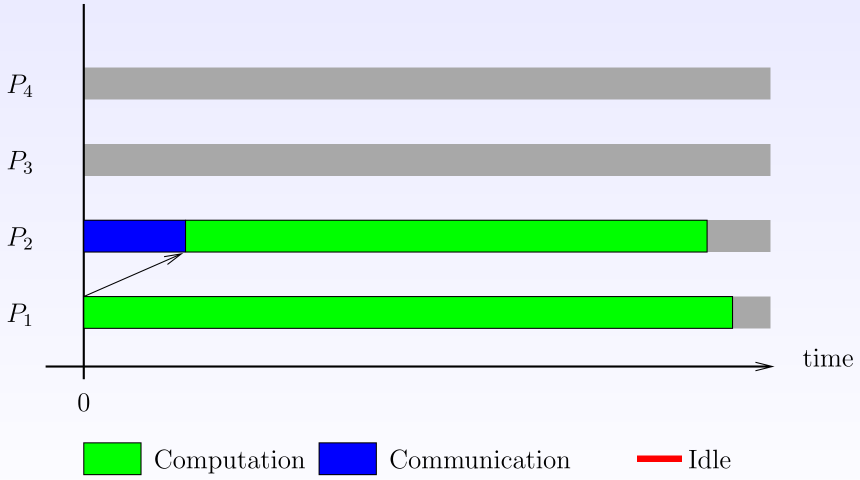

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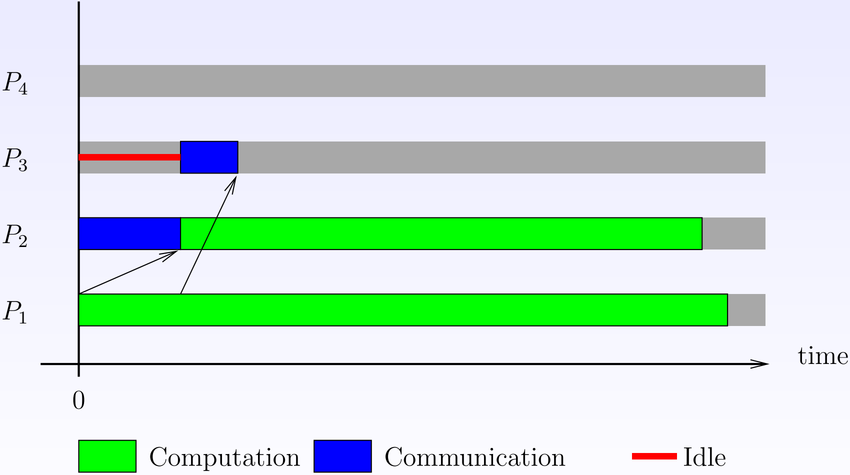

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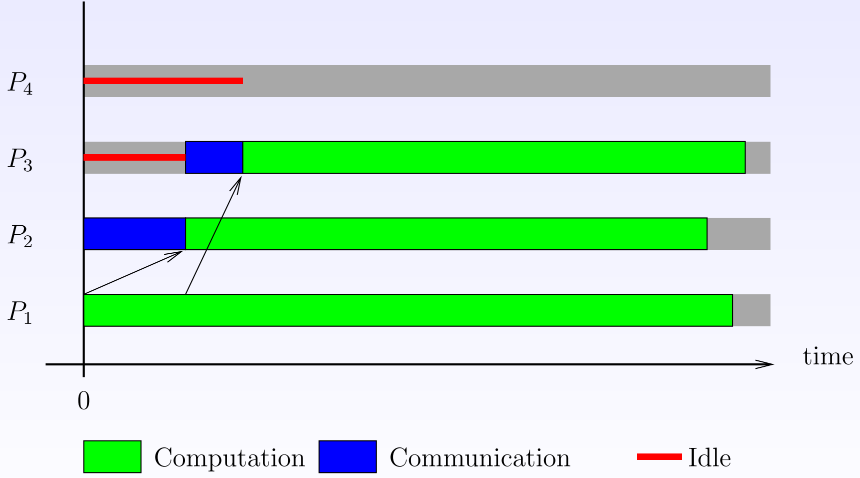

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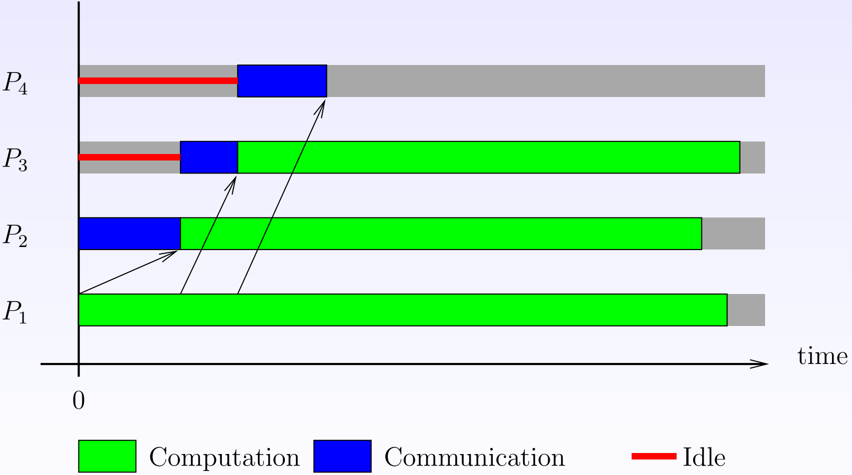

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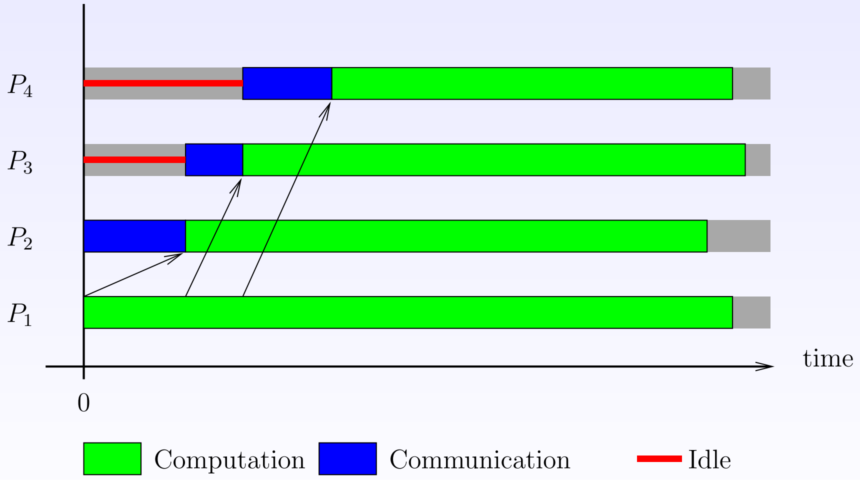

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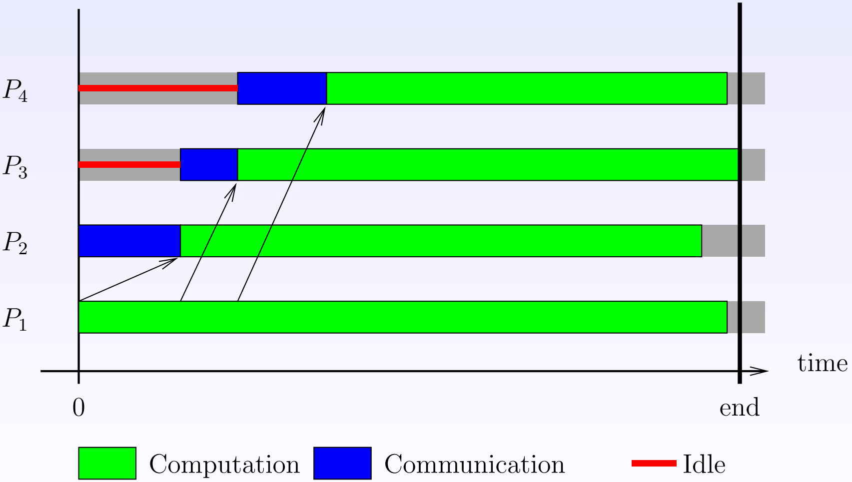

Problem specification

- Master sends n_i data items to P_i within a single message
- Master sequentially sends messages to slaves in the order

$$P_2, P_3, ..., P_p$$

- Simultaneously, master processes its own n_1 data items
- A slave cannot start computing before reception of its message is complete
- Question: compute best allocation and corresponding schedule

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Equations

- P_1: $T_1 = n_1.w_1$
- P_2: $T_2 = n_2.c + n_2.w_2$
- P_3: $T_3 = (n_2.c + n_3.c) + n_3.w_3$
- P_i: $T_i = \sum_{j=2}^{i} n_j.c + n_i.w_i$ for $i \geq 2$
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Execution time (1/2)

$$T = \max_{1 \le i \le p} \left(\sum_{j=1}^{i} n_j.c_j + n_i.w_i \right)$$

Goal: find distribution n_1, ..., n_p to minimize T

Execution time (2/2)

$$T = \max\left(n_1.c_1 + n_1.w_1, \max_{2\leq i\leq p}\left(\sum_{j=1}^{i} n_j.c_j + n_i.w_i\right)\right)$$

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Optimal distribution of W_{total} data among p processors:
- assign n_1 data items to P_1
- use optimal distribution of $W_{\mathsf{total}} - n_1$ data items among $p-1$ remaining processors

Dynamic Programming Algorithm

solution$[0, p] \leftarrow \mathrm{cons}(0, NIL)$; *cost*$[0, p] \leftarrow 0$
for $d \leftarrow 1$ **to** W_{total} **do**
    solution$[d, p] \leftarrow \mathrm{cons}(d, NIL)$
    cost$[d, p] \leftarrow d \cdot c_p + d \cdot w_p$
for $i \leftarrow p - 1$ **downto** 1 **do**
    solution$[0, i] \leftarrow \mathrm{cons}(0,$ *solution*$[0, i + 1])$
    cost$[0, i] \leftarrow 0$
    for $d \leftarrow 1$ **to** W_{total} **do**
        $(sol, min) \leftarrow (0,$ *cost*$[d, i + 1])$
        for $e \leftarrow 1$ **to** d **do**
            $m \leftarrow e \cdot c_i + \max(e \cdot w_i,$ *cost*$[d - e, i + 1])$
            if $m < min$ **then**
                $(sol, min) \leftarrow (e, m)$
        solution$[d, i] \leftarrow \mathrm{cons}(sol,$ *solution*$[d - sol, i + 1])$
        cost$[d, i] \leftarrow min$
return (*solution*$[W_{\text{total}}, 1]$, *cost*$[W_{\text{total}}, 1]$)

Complexity

- **Theoretical**

$$O(W_{\text{total}}^2 \cdot p)$$

- **In practice**
  With $W_{\text{total}} = 817101$ and $p = 16$, using a 933MHz Pentium III: over two days ...
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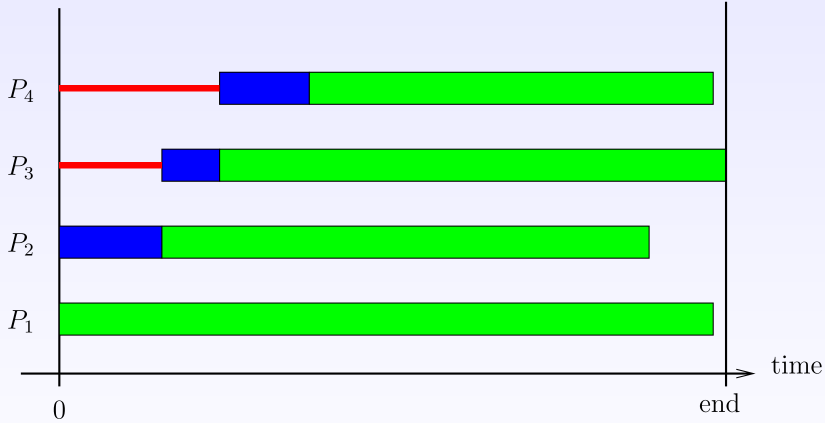
Load balancing property

Lemma

In any optimal solution, all processors terminate execution simultaneously

Proof of Lemma (1/2)

Two slaves i and $i+1$ with $T_i < T_{i+1}$



Decrease α_{i+1} by ε

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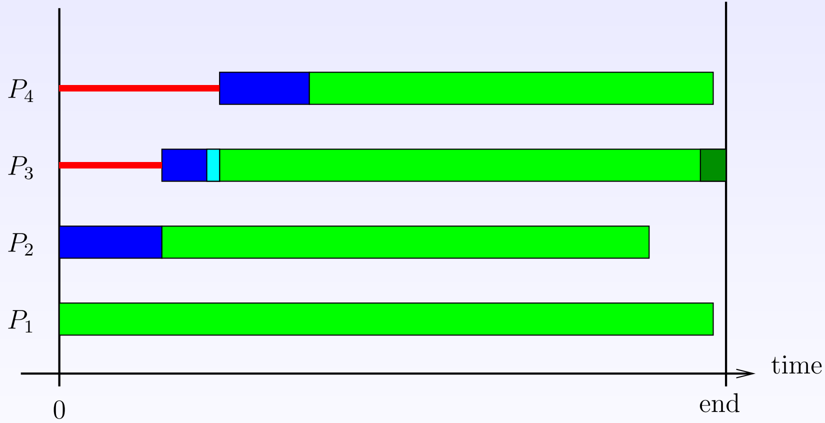

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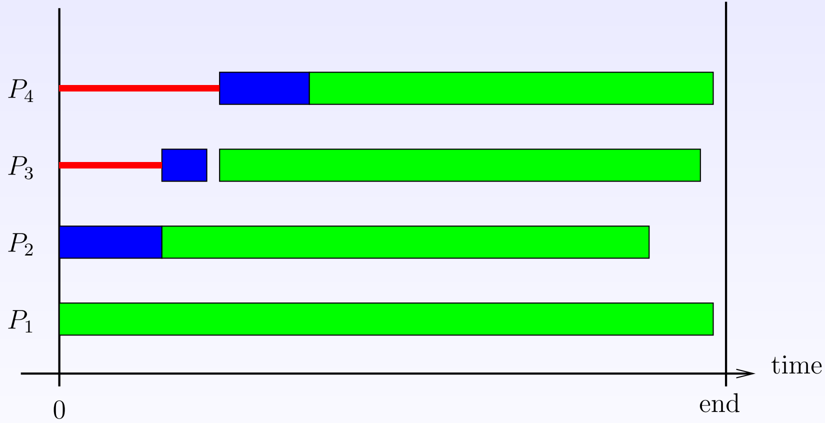

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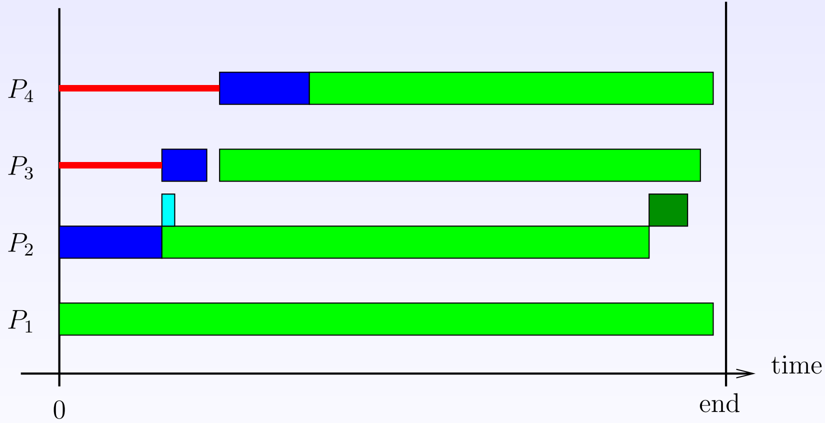

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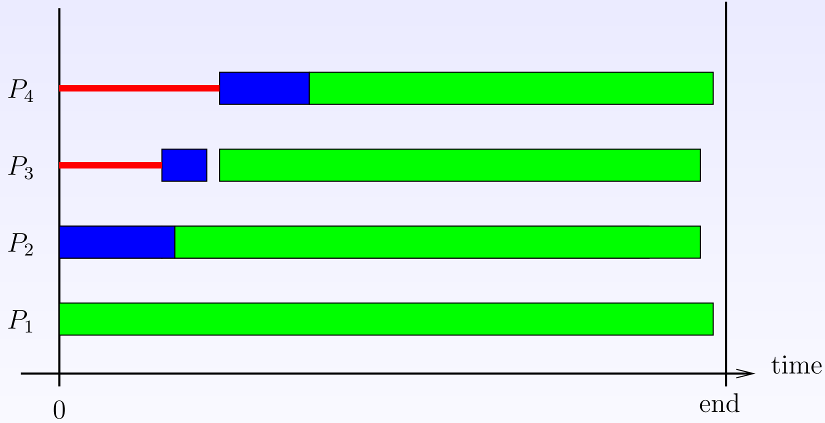

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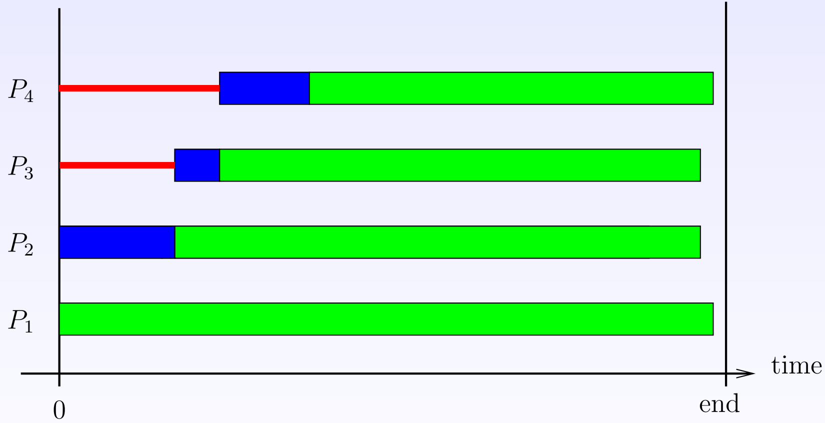

Communication time for following processors does not change

Proof of Lemma (1/2)

Two slaves i and $i+1$ with $T_i < T_{i+1}$

Get better solution!

Proof of Lemma (2/2)

- Ideally: $T'_i = T'_{i+1}$
  Choose ε such that:
  $(\alpha_i+\varepsilon)W_{\mathsf{total}}(c+w_i) = (\alpha_i+\varepsilon)W_{\mathsf{total}}c+(\alpha_{i+1}-\varepsilon)W_{\mathsf{total}}(c+w_{i+1})$
- If master finishes earlier: absurd
- If master finishes later: similar, decrease its load by ε

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Lemma

In any optimal solution, all processors are enrolled

Proof: simple follow-up of previous Lemma

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Load processed by P_i and P_{i+1} within time T

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No Impact!

Choosing master processor

Compare processors P_1 and P_2

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Load maximal when $w_1 < w_2$
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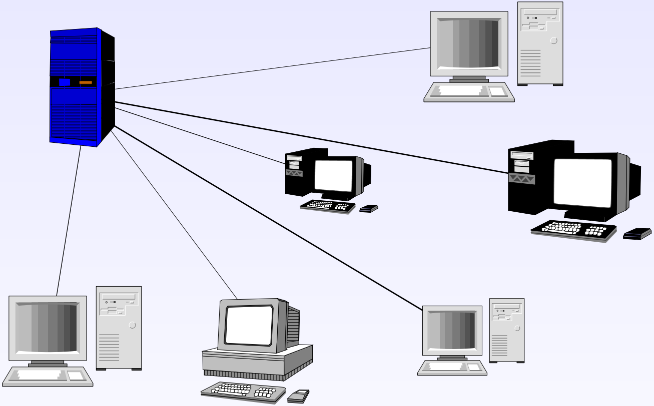
Conclusion

- Closed-form formula for execution time and load distribution
- Choice of master
- Ordering of slaves not important
- All processors are enrolled
- Powerful approach! ☺

Star network



- Communication links between master and slaves have different bandwidths
- Slaves have different computing power

Notations

- Set P_1, ..., P_p of processors
- P_1 is the master, initially holds all data
- Total amount of work: W_{total}
- Processor P_i receives $\alpha_i W_{\text{total}}$ load units,
  where $\alpha_i W_{\text{total}} \in \mathbb{Q}$ and $\sum_i \alpha_i = 1$
  Time for one load unit on P_i: w_i
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$\rightsquigarrow$ **Serve processors with higher bandwidth first**

Resource selection

Lemma

In any optimal solution, all processors are enrolled

Proof:

- Consider an optimal solution
- Let P_k be a non-participating processor
- Enroll P_k in the end and assign load α_k s.t.
- $\alpha_k(c_k + w_k)W_{\text{total}} =$ execution time of last activated processor

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Conclusion

- Closed-form formula for execution time and load distribution
- Serve faster-communicating processors first
- All processors terminate execution simultaneously
- All processors are enrolled
- Powerful approach! ☺

Various extensions

Good news One-round, linear model extends to other topologies (e.g. trees, linear chains)

Still open Adding return messages, although very natural, renders problem quite combinatorial

Bad news

- Becomes NP-hard when adding communication/computation latencies
- Unfortunately, adding latencies absolutely needed to deal with multi-round algorithms

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Outline

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