Scheduling

Lecture 1: Introduction

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Welcome

- Who am I?: CNRS researcher at LIP, used to be a student of ENS Lyon, and a PhD student at LIP.
- Any information about the class:
  loris.marchal@ens-lyon.fr + website (google me)
- Mostlly on the board, but slides will be available on the website
- Outline of the class:
  - today: introduction to scheduling
  - after: study of particular scheduling problems
  - focus on scheduling for large-scale platforms
  - at the end: more on-going research stuff
  - evaluations: research papers
- Slides and documentation source:
  - myself (a little bit)
  - Frédéric Vivien http://graal.ens-lyon.fr/~fvivien/
Outline

Vocabulary

- Basic complex scheduling problem
- Processor scheduling

Graham classification

Types of results: easy and hard problems

Some scheduling problems

1 ∥ ∑ w_i C_i, polynomial (Smith-ratio)

P | prec | C_{max}, NP-hard, Graham 2-approx

1 ∥ ∑ U_i, Moore-Hodgson algorithm

Other types of scheduling problems?
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Other types of scheduling problems?
What is scheduling?

- Allocation of limited resources to activities over time
- **Activities**: tasks in computer environment, steps of a construction project, operations in a production process, lectures at the University, etc.
- **Resources**: processors, workers, machines, lecturers, rooms, etc.

Many variations on the model, on the resource/activity interaction and on the objective.
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Other types of scheduling problems?
Resource-constrained project scheduling problem:

Schedule activities over time on scarce resources, such that some constraints are satisfied and some objective function is optimized.

- $n$ activities (jobs) $j = 1, \ldots, n$,
- $r$ renewable resources $i = 1, \ldots, r$
- $R_k$: amounts of resource $k$ available at any time
- activity $j$ processed for $p_j$ time units, using an amount $r_{j,k}$ of resource $k$
- Integer numbers
- $R_k = 1 \iff$ resource disjunctive ($0 \iff$ resource cumulative)
- Precedence constraints: $i \rightarrow j$
  - $j$ cannot start before $i$ is completed
- Precedence constraint graph: DAG (directed acyclic graph)
Objective: find starting time \( S_j \) for each activity, such that

- At each time, the total resource demand is less than (or equal to) the resource availability for each resource
- Precedence constraints are satisfied: \( S_i + p_i \leq S_j \) if \( i \rightarrow j \)
- conceptMakespan \( C_{\text{max}} = \max C_j \) is minimized, with \( C_j = S_j + p_j \)

- \( C_j = S_j + p_j \) implies no preemption (activity splitting).
- Dummy starting activity \( 0 \) + dummy termination activity \( n + 1 \), with \( S_0 = 0 \) and \( C_{\text{max}} = C_{n+1} \)
- Without preemption, vector \( S \) defines a schedule
- \( S \) is called feasible if all resource and precedence constraints are fulfilled
Basic complex scheduling problem – Example

<table>
<thead>
<tr>
<th>j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_j$</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>$r_{j,1}$</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$r_{j,2}$</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

(a) A feasible schedule

(b) An optimal schedule
Generalization of precedence relations

- Generalize precedence relation: \( S_i + d_{i,j} \leq S_j \)
  - different cases + models all relations between start/finish times

- Release times \( r_j \) and deadlines
  \[ d_j: \ S_0 + r_j \leq S_j, \ S_j - (d_j - p_j) \leq S_0 \]

- Communication delays \( c_{i,j} \)

(a) positive time-lag

(b) negative time-lag
Other objectives

- **Total flow time**: $\sum_{j=1}^{n} C_j$
- **Weighted (total) flow time**: $\sum_{j=1}^{n} w_j C_j$
- **With due dates $d_j$**:
  - **lateness**: $L_j = C_j - d_j$
  - **tardiness**: $T_j = \max\{0, C_j - d_j\}$
  - **unit penalty**: $U_j = 0$ if $C_j \leq d_j$, 1 otherwise
- **maximum lateness**: $L_{max} = \max L_j$
- **total tardiness**: $\sum T_j$
- **total weighted tardiness**: $\sum w_j T_j$
- **number of late activities**: $\sum U_j$
- **weighted number of late activities**: $\sum w_j U_j$
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\[ 1|| \sum w_i C_i, \text{ polynomial (Smith-ratio)} \]
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\[ 1|| \sum U_i, \text{ Moore-Hodgson algorithm} \]

Other types of scheduling problems?
Single processor scheduling

- $n$ jobs $J_1, \ldots, J_n$ with processing times $p_j$
- 1 processor
- precedence constraint

Example:
- 5 jobs, with processing times 3, 2, 4, 2, 5
- precedence constraints: 1 $\rightarrow$ 3, 2 $\rightarrow$ 4, 4 $\rightarrow$ 5
Parallel processor scheduling

- $m$ identical processors $P_1, \ldots, P_m$
- all tasks have the same processing time $P_j$ on all processors
- $\iff$ RCPSP with one machine, $R_1 = m$

![Diagram]

Variants:
- unrelated processors: $p_{j,k}$ depends on $P_k$ and $J_j$
- uniform processors: $p_{j,k} = P_j / s_k$, $s_k$ is the speed of processor $P_k$
Multi-processor task scheduling

- jobs $J_1, \ldots, J_n$
- processors $P_1, \ldots, P_m$
- each job $J_j$ has processing time $p_j$, and makes use of a subset of processor $\mu_j \subseteq \{P_1, \ldots, P_m\}$
- + precedence constraints

another variant: identical processors, and each job $J_j$ makes use of any subset of $size_j$ processors
Shop scheduling

Jobs consist in several operations, to be processed on different resources.

General shop scheduling problem:

- jobs $J_1, \ldots, J_n$
- processors $P_1, \ldots, P_m$
- $J_j$ consists in $n_j$ operation $O_{1,j}, \ldots, O_{n_j,j}$
- two operations of the same job cannot be processed at the same time
- a processor can process one operation at a time
- operations $O_{i,j}$ has processing time $p_{i,j}$ and makes use of processor $\mu_{i,j}$
- arbitrary precedence pattern
Shop scheduling

job-shop scheduling problem:
  ▶ chain of precedence constraints:

\[ O_{1,j} \rightarrow O_{2,j} \rightarrow \cdots \rightarrow O_{n_{j},j} \]

flow-shop scheduling problem:
  ▶ special job-shop scheduling problem
  ▶ \( n_{j} = m \) for all \( j \), and \( \mu_{i,j} = P_{i} \) for all \( i, j \):
    operation \( O_{i,j} \) must be processed by \( P_{i} \)

open-shop scheduling problem
  ▶ like a flow-shop, but no precedence constraints
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Other types of scheduling problems?
Classes of scheduling problems can be specified in terms of the three-field classification $\alpha|\beta|\gamma$ where

- $\alpha$ specifies the machine environment,
- $\beta$ specifies the job characteristics,
- $\gamma$ and describes the objective function(s).
Graham notation – machines

To describe the machine environment the following symbols are used:

- ▶ 1 single machine
- ▶ P parallel identical
- ▶ Q uniform machines
- ▶ R unrelated machines
- ▶ MPM multi-purpose machines
- ▶ J job-shop
- ▶ F flow-shop
- ▶ O open-shop

The above symbols are used if the number of machines is part of the input. If the number of machines is fixed to $m$ we write $P_m$, $Q_m$, $R_m$, $MPM_m$, $J_m$, $F_m$, $O_m$. 
Graham notation – Job characteristics

- pmtm preemption
- $r_j$ release times
- $d_j$ deadlines
- $p_j = 1$ or $p_j = p$ or $p_j \in 1, 2$: restricted processing times
- prec: arbitrary precedence constraints
- intree: (outtree) intree (or outtree) precedences
- chains: chain precedences
- series-parallel: a series-parallel precedence graph
Graham notation – Objectives

- makespan $C_{\text{max}}$
- maximum lateness: $L_{\text{max}}$
- mean flow-time $\sum C_i$
- mean weighted flow-time $\sum w_i C_i$

- sum of tardiness $\sum T_j$
- sum of weighted tardiness $\sum w_j T_j$

- number of late jobs $\sum U_j$
- weighted number of late activities $\sum w_j U_j$

( lateness: $L_j = C_j - d_j$, tardiness: $T_j = \max\{0, C_j - d_j\}$, unit penalty: $U_j = 0$ if $C_j \leq d_j$, 1 otherwise)
1\mid r_j; pmtn\mid L_{max}

P2\mid p_j = p; r_j; tree\mid C_{max}

Jm\mid p_{i,j} = 1\mid \sum w_j U_j
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Other types of scheduling problems?
Polynomial problems

- a solution to a scheduling problem is a function $h$:
  - $x$ is the input (parameters)
  - $h(x)$ is the solution (starting times, etc.)
- $|x|$ defined as the length of some encoding of $x$
  - usually, binary encoding: integer $a$ encoded in $\log_2 a$ bits
- Complexity of an algorithm computing $h(x)$ for all $x$: running time
- An algorithm is called polynomial, if it computes $h(x)$ for all $x$ it at most $O(P(|x|))$ steps, where $P$ is a polynomial
- A problem is called polynomial if it can be solved by a polynomial algorithm
Pseudo-polynomial problems

If we replace the binary encoding by an unary encoding (integer $a$ encoded with size $a$): we can solve more difficult problems in time polynomial with $|x|$.

- An algorithm is pseudo-polynomial if it solves the problem for all $x$ with a number of steps at most $O(p(|x|))$ steps, where $P$ is a polynomial and $|x|$ the size of an unary encoding of $x$.

Example:
An algorithm for a scheduling problem, whose running time is $O(p_j)$ is pseudo-polynomial.
P and NP

- Decision problems
- To each optimization problem, we can define a decision problem
- P: class of polynomially solvable decision problems
- NP: class of polynomially checkable decision problems for each "yes"-answer, a certificate exists which can be used to check the answer in polynomial time
- Decisions problems of scheduling problems belongs to NP
- $P \subseteq NP$. $P \neq NP$ still open
a decision problem Q is NP-complete if all problems in NP can be polynomially reduced to Q

if any single NP-complete decision problem Q could be solved in polynomial time then we would have $P = NP$.

To prove that a problem is NP-complete: reduction to a well-known NP-complete problem

Weakly NP-complete (or binary NP-complete): strongly depends on the binary coding of the input. If unary coding is used, the problem might become polynomial (pseudo-polynomial).

- 2-Partition vs 3-Partition
How to solve NP-complete problems?

Exact methods:
- Mixed integer linear programming
- Dynamic programming
- Branch and bound methods
  (usually limited to small instances)

Approximate methods:
- Heuristics (no guarantee)
- Approximation algorithms
Approximation algorithms

Consider a minimization problem. On a given instance $x$, $f(x)$: value of the objective in the solution given by the algorithm $f^*(x)$: optimal value of the objective

An algorithm is a $\rho$-approximation if for any instance $x$, $f(x) \leq \rho \times f^*(x)$

APX class: problems for which there exists a polynomial-time $\rho$-approximation algorithm, for some $\rho > 0$

An algorithm is a PTAS (Polynomial Time Approximation Scheme) if for any instance $x$ and any $\epsilon > 0$, the algorithm computes a solution $f(x)$ with $f(x) \leq (1 + \epsilon) \times f^*(x)$ in time polynomial in the problem size.

An algorithm is a FPTAS (Fully Polynomial Time Approximation Scheme) if for any instance and any $\epsilon > 0$, it produces a solution $f(x)$ such that $f(x) \leq (1 + \epsilon) \times f^*(x)$, in time polynomial in the problem size and in $1/\epsilon$. 
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Other types of scheduling problems?
Objective: weighted sum of completion times

Intuitions:
- put high weight first
- put longer tasks last

⇒ Order task by non-increasing Smith ratio:

\[ \frac{w_1}{p_1} \geq \frac{w_2}{p_2} \geq \cdots \geq \frac{w_n}{p_n} \]

Proof:
- Consider a different optimal schedule \( S \)
- Let \( i \) and \( j \) be two consecutive tasks in this schedule such that \( \frac{w_i}{p_i} < \frac{w_j}{p_j} \)

contribution of these tasks in \( S \):
\[ S_i = (w_i + w_j)(t + p_i) + w_j p_j \]

collection of these tasks if switched:
\[ S_j = (w_i + w_j)(t + p_j) + w_i p_i \]

we have
\[ \frac{S_i - S_j}{w_i w_j} = \frac{p_i}{w_i} - \frac{p_j}{w_j} \]

Thus we decrease the objective by switching these tasks.
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Other types of scheduling problems?
$P|\text{prec}|C_{\text{max}}$, Graham 2-approx

- NP-complete
- reduction to 2-partition (or 3-partition, $\rightarrow$ unary NP-complete)
Graham list scheduling approximation

- **Theorem:** Any list scheduling heuristic gives a schedule, whose makespan is at most $2 - 1/p$ times the optimal.

- **Lemma:** there exists a precedence path $\Psi$ such that 
  \[ \text{Idle} \leq (p - 1) \times w(\Psi) \]
  - Consider the task with maximum termination time $T_1$
  - Let $t_1$ be the last moment (strictly) before $\sigma(T_1)$ when a processor is not active
  - Since a processor is inactive at time $t_1$, there exists a task $T_2$, finishing at time $t_1$, which is an ancestor of $T_1$ (unless $T_1$ would be free and scheduled at time $t_1$ or before)
  - Iterate the process
  - All idle times occur during the processing of these tasks, at most on $p - 1$ processors

- Notice that $pC_{\text{max}} = \text{Idle} + \text{Seq}$, with $\text{Seq} = \sum w(T_i)$

- We also have $\text{Seq} \leq pC_{\text{max}}^{\text{opt}}$, thus
  \[ C_{\text{max}} \leq ((p - 1) \times w(\Psi)) + (pC_{\text{max}}^{\text{opt}}) \]

- We also have $w(\Psi) \leq C_{\text{max}}^{\text{opt}}$, qed.
The approximation bound is tight

\[ T_{1}^{(K(p-1))} \quad T_{2}^{(K(p-1))} \quad \ldots \quad T_{p-1}^{(K(p-1))} \]

\begin{align*}
C_{\text{list}}^{\max} &= Kp + K(p-1) = K(2p-1) \\
C_{\text{opt}}^{\max} &= 1 + K + K(p-1) = Kp + 1
\end{align*}

\[
\frac{C_{\text{list}}^{\max}}{C_{\text{opt}}^{\max}} \geq \frac{K(2p-1)}{Kp+1} = \frac{2p-1}{p} - \frac{2p-1}{p(Kp+1)} = \left(2 - \frac{1}{p}\right) - \epsilon(K),
\]
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Other types of scheduling problems?
One machine, minimize the number of late jobs

Example:

<table>
<thead>
<tr>
<th>job</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_j$</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>$p_j$</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

Tasks are sorted by non-decreasing $d_i : d_1 \leq \cdots \leq d_n$

- $A := \emptyset$
- For $i = 1 \ldots n$
  - If $p(A) + p_i \leq d_i$, then $A := A \cup \{i\}$
  - Otherwise,
    - Let $j$ be the longest task in $A \cup \{i\}$
    - $A := A \cup \{i\} - \{j\}$

Optimal solution : $A = \{2, 3, 5\}$
Feasibility

We first prove that the algorithm produces a feasible schedule:

- By induction: if not task is rejected, ok
- Assume that $A$ is feasible, prove that $A \cup \{i\} - \{j\}$ is feasible too
  - all tasks in $A$ before $j$: no change
  - all tasks in $A$ after $j$: shorter completion
  - task $i$: let $k$ be the last task in $A$: $p(A) \leq d_k$
    since task $j$ is the longest: $p_i \leq p_j$, thus
    $p \cup \{i\} - \{j\} \leq p(A) \leq d_k \leq d_i$ (because tasks are sorted)
Assume that there exist an optimal set $O$ different from the set $A_f$ output by the Moore-Hodgson algorithm

- Let $j$ be the first task rejected by the algorithm
- We prove that there exists an optimal solution without $j$
- We consider the set $A = \{1, \ldots, i - 1\}$ at the moment when task $j$ is rejected from $A$, and $i$ the task being added at this moment
- $A + i$ is not feasible, thus $O$ does not contain $\{1, \ldots, i\}$
- Let $k$ be a task of $\{1, \ldots, i\}$ which is not in $O$
- Since the algorithm rejects the longest task, $p(O \cup \{k\} - \{j\}) \leq p(O)$, and by the same arguments than before, $O \cup \{k\} - \{j\}$ is feasible
- We can suppress $j$ from the problem instance, without modifying the behavior of the algorithm or the objective

We can repeat this process, until we get the set of tasks scheduled by the algorithm.
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Other types of scheduling problems ?
Other types of scheduling problems

► Online problems
  ▶ contrarily to offline, information about future jobs is not known in advance
  ▶ competitive ratio: ratio to the optimal offline algorithm

► Distributed scheduling
  ▶ use only local information

► Multi-criteria scheduling
  ▶ several objectives to optimize simultaneously
  ▶ and/or several users, link with game theory

► Cyclic scheduling
  ▶ infinite but regular pattern of tasks