Scheduling
Lecture 1: Introduction

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Welcome

- Who am I ?: CNRS researcher at LIP, used to be a student of ENS Lyon, and a PhD student at LIP.
- Any information about the class: `loris.marchal@ens-lyon.fr` + website (google me)
- Mosty on the board, but slides will be available on the website
- Outline of the class:
  - today: introduction to scheduling
  - after: study of particular scheduling problems
  - focus on scheduling for large-scale platforms
  - at the end: more on-going research stuff
  - evaluations: research papers
- Slides and documentation source:
  - myself (a little bit)
  - Frédéric Vivien `http://graal.ens-lyon.fr/~fvivien/`
  - EPIT school `http://graal.ens-lyon.fr/~fvivien/EPIT2007.html`, forthcoming book

Outline

- Vocabulary
  - Basic complex scheduling problem
  - Processor scheduling
- Graham classification
- Types of results: easy and hard problems
- Some scheduling problems
  - $1||\sum w_i C_i$, polynomial (Smith-ratio)
  - $P|prec|C_{\max}$, NP-hard, Graham 2-approx
  - $1||\sum U_i$, Moore-Hodgson algorithm
- Other types of scheduling problems ?

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Other types of scheduling problems ?

What is scheduling ?

- allocation of limited resources to activities over time
- activities: tasks in computer environment, steps of a construction project, operations in a production process, lectures at the University, etc.
- resources: processors, workers, machines, lecturers, rooms, etc.

Many variations on the model, on the resource/activity interaction and on the objective.

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Other types of scheduling problems ?

Basic complex scheduling problem

Resource-constrained project scheduling problem:
Schedule activities over time on scarce resources, such that some constraints are satisfied and some objective function is optimized

- n activities (jobs) $j = 1, \ldots n$,
- r renewable resources $i = 1, \ldots, r$
- R_k: amounts of resource k available at any time
- activity j processed for p_j time units,
  using an amount $r_{j,k}$ of resource k
- Integer numbers
- $R_k = 1 \Leftrightarrow$ resource disjunctive (0 $\Leftrightarrow$ resource cumulative)
- Precedence constraints: $i \rightarrow j$
  - j cannot start before i is completed
- Precedence constraint graph: DAG (directed acyclic graph)

Basic complex scheduling problem

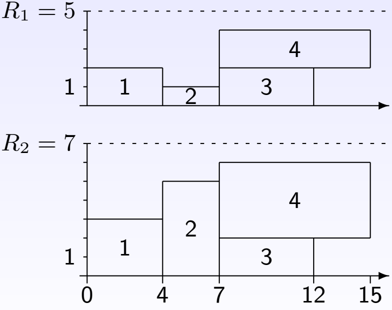
Objective: find starting time S_j for each activity, such that

- At each time, the total resource demand is less than (or equal to) the resource availability for each resource
- Precedence constraints are satisfied: $S_i + p_i \leq S_j$ if $i \rightarrow j$
- conceptMakespan $C_{\max} = \max C_j$ is minimized, with $C_j = S_j + p_j$

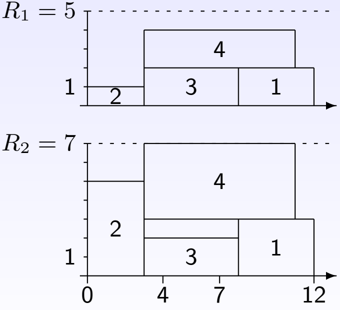
- $C_j = S_j + p_j$ implies no preemption (activity splitting).
- Dummy starting activity 0 + dummy termination activity $n+1$, with $S_0 = 0$ and $C_{\max} = C_{n+1}$
- Without preemption, vector S defines a schedule
- S is called feasible if all resource and precedence constraints are fulfilled

Basic complex scheduling problem – Example

j	1	2	3	4
p_j	4	3	5	8
$r_{j,1}$	2	1	2	2
$r_{j,2}$	3	5	2	4



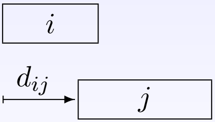
(a) A feasible schedule



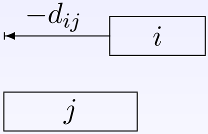
(b) An optimal schedule

Generalization of precedence relations

- Generalize precedence relation: $S_i + d_{i,j} \leq S_j$
  - different cases + models all relations between start/finish times
- Release times r_j and deadlines d_j: $S_0 + r_j \leq S_j$, $S_j - (d_j - p_j) \leq S_0$
- Communication delays $c_{i,j}$



(a) positive time-lag



(b) negative time-lag

Other objectives

- Total flow time: $\sum_{j=1}^{n} C_j$
- Weighted (total) flow time: $\sum_{j=1}^{n} w_j C_j$
- With due dates d_j:
  - lateness: $L_j = C_j - d_j$
  - tardiness: $T_j = \max\{0, C_j - d_j\}$
  - unit penalty: $U_j = 0$ if $C_j \leq d_j$, 1 otherwise
- maximum lateness: $L_{mathrmmax} = \max L_j$
- total tardiness $\sum T_j$
- total weighted tardiness $\sum w_j T_j$
- number of late activities $\sum U_j$
- weighted number of late activities $\sum w_j U_j$

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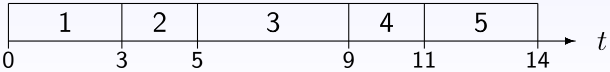
Other types of scheduling problems ?

Single processor scheduling

- n jobs $J_1, \ldots, J_n$ with processing times p_j
- 1 processor
- precedence constraint

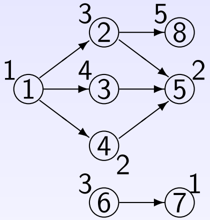
Example:

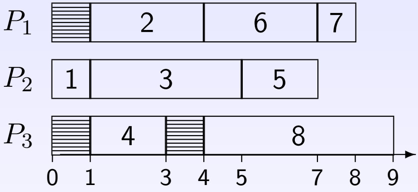
- 5 jobs, with processing times $3, 2, 4, 2, 5$
- precedence constraints: $1 \to 3, 2 \to 4, 4 \to 5$



Parellel processor scheduling

- m identical processors $P_1, \ldots, P_m$
- all tasks have the same processing time P_j on all processors
- $\Leftrightarrow$ RCPSP with one machine, $R_1 = m$





Variants:

- unrelated processors: $p_{j,k}$ depends on P_k and J_j
- uniform processors: $p_{j,k} = P_j / s_k$, s_k is the speed of processor P_k

Multi-processor task scheduling

- jobs $J_1, \dots J_n$
- processors $P_1, \dots, P_m$
- each job J_j has processing time p_j, and makes use of a subset of processor $\mu_j \subseteq \{P_1, \dots, P_m\}$
- $+$ precedence constraints

another variant: identical processors, and each job J_j makes us of any subset of $size_j$ processors

Shop scheduling

Jobs consist in several operations, to be processed on different resources.

General shop scheduling problem:

- jobs $J_1, \dots J_n$
- processors $P_1, \dots, P_m$
- J_j consists in n_j operation $O_{1,j}, \dots, O_{n_j,j}$
- two operations of the same job cannot be processed at the same time
- a processor can process one operation at a time
- operations $O_{i,j}$ has processing time $p_{i,j}$ and makes use of processor $\mu_{i,j}$
- arbitrary precedence pattern

Shop scheduling

job-shop scheduling problem:

- chain of precedence constraints:

$$O_{1,j} \to O_{2,j} \to \cdots \to O_{n_j,j}$$

flow-shop scheduling problem:

- special job-shop scheduling problem
- $n_j = m$ for all j, and $\mu_{i,j} = P_i$ for all i, j:
  operation $O_{i,j}$ must be processed by P_i

open-shop scheduling problem

- like a flow-shop, but no precedence constraints

Outline

Graham notation

Classes of scheduling problems can be specified in terms of the three-field classification $\alpha|\beta|\gamma$ where

- α specifies the machine environment,
- β specifies the job characteristics,
- γ and describes the objective function(s).

Graham notation – machines

To describe the machine environment the following symbols are used:

- 1 single machine
- P parallel identical
- Q uniform machines
- R unrelated machines
- MPM multi-purpose machines
- J job-shop
- F flow-shop
- O open-shop

The above symbols are used if the number of machines is part of the input. If the number of machines is fixed to m we write Pm, Qm, Rm, MPMm, Jm, Fm, Om.

Graham notation – Job characteristics

- pmtn preemption
- r_j release times
- d_j deadlines
- $p_j = 1$ or $p_j = p$ or $p_j \in 1, 2$: restricted processing times
- prec : arbitrary precedence constraints
- intree: (outtree) intree (or outtree) precedences
- chains: chain precedences
- series-parallel: a series- parallel precedence graph

Graham notation – Objectives

- makespan $C_{\max}$
- maximum lateness: $L_{\max}$
- mean flow-time $\sum C_i$
- mean weighted flow-time $\sum w_i C_i$

- sum of tardiness $\sum T_j$
- sum of weighted tardiness $\sum w_j T_j$

- number of late jobs $\sum U_j$
- weighted number of late activities $\sum w_j U_j$

(lateness: $L_j = C_j - d_j$, tardiness: $T_j = \max\{0, C_j - d_j\}$, unit penalty: $U_j = 0$ if $C_j \leq d_j$, 1 otherwise)

Graham notation – Examples

$$1|r_j; pmtn|L_{\max}$$

$$P2|p_j = p; r_j; tree|C_{\max}$$

$$Jm|p_{i,j} = 1|\sum w_j U_j$$

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Other types of scheduling problems ?

Polynomial problems

- a solution to a scheduling problem is a function h:
  - x is the input (parameters)
  - $h(x)$ is the solution (starting times, etc.)
- $|x|$ defined as the length of some encoding of x
  - usually, binary encoding: integer a encoded in $\log_2 a$ bits
- Complexity of an algorithm computing $h(x)$ for all x: running time
- An algorithm is called polynomial, if it computes $h(x)$ for all x it at most $O(p(|x|)$ steps, where P is a polynomial
- A problem is called polynomial if it can be solved by a polynomial algorithm

Pseudo-polynomial problems

If we replace the binary encoding by an unary encoding (integer a encoded with size a): we can solve more difficult problems in time polynomial with $|x|$.

- An algorithm is pseudo-polynomial if it solves the problem for all x with a number of steps at most $O(p(|x|)$ steps, where P is a polynomial and $|x|$ the size of of an unary encoding of x.

Example:
An algorithm for a scheduling problem, whose running time is $O(p_j)$ is pseudo-polynomial.

P and NP

- Decision problems
- To each optimization problem, we can define a decision problem
- P: class of polynomially solvable decision problems
- NP: class of polynomially *checkable* decision problems
  for each "yes"-answer, a certificate exists which can be used to check the answer in polynomial time
- Decisions problems of scheduling problems belongs to NP
- $P \subseteq NP$. $P \stackrel{?}{=} NP$ still open

NP-complete problems

- a decision problem Q is NP-complete if all problems in NP can be polynomially reduced to Q
- if any single NP-complete decision problem Q could be solved in polynomial time then we would have P = NP.
- To prove that a problem is NP-complete: reduction to a well-known NP-complete problem

- Weakly NP-complete (or binary NP-complete): strongly depends on the binary coding of the input. If unary coding is used, the problem might become polynomial (pseudo-polynomial).
  - 2-Partition vs 3-Partition

How to solve NP-complete problems ?

Exact methods:

- Mixed integer linear programming
- Dynamic programming
- Branch and bound methods

(usually limited to small instances)

Approximate methods:

- Heuristics (no guarantee)
- Approximation algorithms

Approximation algorithms

Consider a minimization problem. On a given instance x,
$f(x)$: value of the objective in the solution given by the algorithm
$f^*(x)$: optimal value of the objective

An algorithm is a ρ-approximation if for any instance x ,
$f(x) \leq \rho \times f^*(x)$

APX class: problems for which there exists a polynomial-time ρ-approximation algorithm, for some $\rho > 0$

An algorithm is a PTAS (Polynomial Time Approximation Scheme) if for any instance x and any $\epsilon > 0$, the algorithm computes a solution $f(x)$ with $f(x) \leq (1 + \epsilon) \times f^*(x)$ in time polynomial in the problem size.

An algorithm is a FPTAS (Fully Polynomial Time Approximation Scheme) if for any instance and any $\epsilon > 0$, it produces a solution $f(x)$ such that $f(x) \leq (1 + \epsilon) \times f^*(x)$, in time polynomial in the problem size and in 1ϵ.

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Other types of scheduling problems ?

$1||\sum w_i C_i$, or the Smith-ratio

- Objective: weighted sum of completion times
- Intuitions:
  - put high weight first
  - put longer tasks last
- $\Rightarrow$ Order task by non-increasing Smith ratio: $w_1/p_1 \geq w_2/p_2 \geq \cdots \geq w_n/p_n$

Proof:

- Consider a different optimal schedule S
- Let i and j be two consecutive tasks in this schedule such that $w_i/p_i < w_j/p_j$
- contribution of these tasks in S: $S_i = (w_i + w_j)(t + p_i) + w_j p_j$
- contribution of these tasks if switched: $S_j = (w_i + w_j)(t + p_j) + w_i p_i$
- we have $\frac{S_i - S_j}{w_i w_j} = \frac{p_i}{w_i} - \frac{p_j}{w_j}$
  Thus we decrease the objective by switching these tasks.

Outline

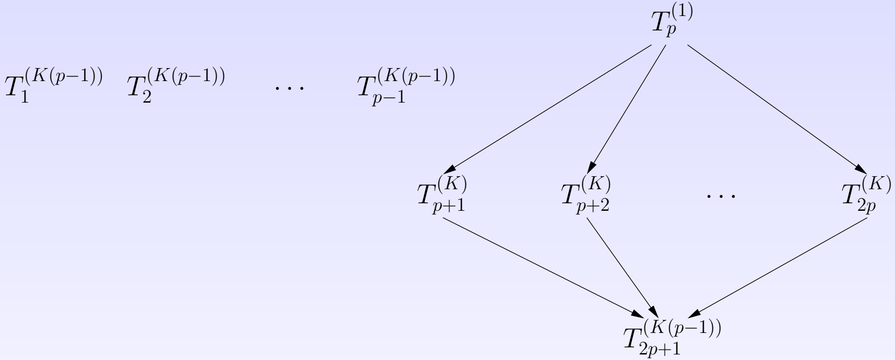
$P|prec|C_{\max}$, Graham 2-approx

- NP-complete
- reduction to 2-partition (or 3-partition, $\longrightarrow$ unary NP-complete)

Graham list scheduling approximation

- Theorem: Any list scheduling heuristic gives a schedule, whose makespan is at most $2 - 1/p$ times the optimal.
- Lemma: there exists a precedence path Ψ such that $\mathrm{Idle} \leq (p-1) \times w(\Psi)$
  - Consider the task with maximum termination time T_1
  - Let t_1 be the last moment (strictly) before $\sigma(T_1)$ when a processor is not active
  - Since a processor is inactive at time t_1, there exists a task T_2, finishing at time t_1, which is an ancestor of T_1 (unless T_1 would be free and scheduled at time t_1 or before)
  - Iterate the process
  - All idle times occur during the processing of these tasks, at most on $p-1$ processors
- Notice that $pC_{\max} = \mathrm{Idle} + \mathrm{Seq}$, with $\mathrm{Seq} = \sum w(T_i)$
- We also have $\mathrm{Seq} \leq pC_{\max}^{\mathrm{opt}}$, thus $C_{\max} \leq \big((p-1) \times w(\Psi)\big) + \big(pC_{\max}^{\mathrm{opt}}\big)$
- We also have $w(\Psi) \leq C_{\max}^{\mathrm{opt}}$, qed.

The approximation bound is tight



$$C_{\max}^{list} = Kp + K(p-1) = K(2p-1)$$
$$C_{\max}^{opt} = 1 + K + K(p-1) = Kp + 1$$

$$\frac{C_{\max}^{list}}{C_{\max}^{opt}} \geqslant \frac{K(2p-1)}{Kp+1} = \frac{2p-1}{p} - \frac{2p-1}{p(Kp+1)} = (2 - \frac{1}{p}) - \epsilon(K),$$

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$1||\sum U_i$, Moore-Hodgson algorithm

One machine, minimize the number of late jobs

Example:

job	1	2	3	4	5
d_j	6	7	8	9	11
p_j	4	3	2	5	6

Tasks are sorted by non-decreasing d_i : $d_1 \leq \cdots \leq d_n$

- $A := \emptyset$
- For $i = 1 \ldots n$
  - $If p(A) + p_i \leq d_i$, then $A := A \cup \{i\}$
  - Otherwise,
    - Let j be the longest task in $A \cup \{i\}$
    - $A := A \cup \{i\} - \{j\}$

Optimal solution : $A = \{2, 3, 5\}$

Feasibility

We first prove that the algorithm produces a feasible schedule:

- By induction: if not task is rejected, ok
- Assume that A is feasible, prove that $A \cup \{i\} - \{j\}$ is feasible too
  - all tasks in A before j: no change
  - all tasks in A after j: shorter completion
  - task i: let k be the last task in A: $p(A) \leq d_k$
    since task j is the longest: $p_i \leq p_j$, thus
    $p \cup \{i\} - \{j\} \leq p(A) \leq d_k \leq d_i$ (because tasks are sorted)

Optimality

Assume that there exist an optimal set O different from the set A_f output by the Moore-Hodgson algorithm

- Let j be the first task rejected by the algorithm
- We prove that there exists an optimal solution without j
- We consider the set $A = \{1, \dots, i-1\}$ at the moment when task j is rejected from A, and i the task being added at this moment
- $A + i$ is not feasible, thus O does not contain $\{1, \dots, i\}$
- Let k be a task of $\{1, \dots, i\}$ which is not in O
- Since the algorithm rejects the longest task, $p(O \cup \{k\} - \{j\}) \leq p(O)$, and by the same arguments than before, $O \cup \{k\} - \{j\}$ is feasible
- We can suppress j from the problem instance, without modifying the behavior of the algorithm or the objective

We can repeat this process, until we get the set of tasks scheduled by the algorithm.

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Other types of scheduling problems

- Online problems
  - contrarily to offline, information about future jobs is not known in advance
  - competitive ratio: ratio to the optimal offline algorithm
- Distributed scheduling
  - use only local information
- Multi-criteria scheduling
  - several objectives to optimize simultaneously
  - and/or several users, link with game theory
- Cyclic scheduling
  - infinite but regular pattern of tasks