Online scheduling

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Outline

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Offline vs. online algorithmics

Nature of the problem

Known

Objective function

Known

Characteristics of the instance

Known beforehand

Offline

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Known beforehand	Discovered during execution
	Characteristics of a job discovered When the job is released
Offline	(Clairvoyant) Online

Offline vs. online algorithmics

Nature of the problem

Known

Objective function

Known

Characteristics of the instance

Known beforehand	Discovered during execution	
	Characteristics of a job discovered	
	When the job is released	When the job completes
Offline	(Clairvoyant) Online	Non-clairvoyant online

Notation and hypotheses

Notation

- Jobs J_1, ..., J_n
  Job J_j arrives in the system at date r_j
  Job J_j has a weight (or a priority) w_j
  Job J_j has an execution time p_j
  Δ is the ratio of the largest to the shortest execution time
- Completion time of job J_j: C_j
  Flow of job J_j : $F_j = C_j - r_j$ (time spent in the system)

Hypotheses

- Job may be preempted
- One machine (1 | *pmtn* | ???)

Evaluating the quality of an online schedule

An online algorithm has a competitive factor ρ if and only if

Whatever the set of jobs T_1, ..., T_n

Online schedule cost$(T_1, ..., T_N) \leq$
$\rho \times$ Optimal off-line schedule cost$(T_1, ..., T_N)$

What should we optimize ?

- Makespan: $\max_j C_j$

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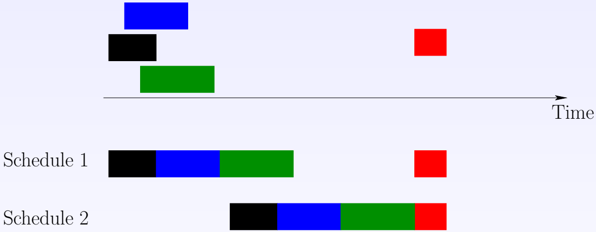

What should we optimize ?

- Makespan: $\max_j C_j$
  Arrival dates are not taken into account
- Average flow or response time: $\sum_j (C_j - r_j)$
  Inconvenient: starvation
- Maximum flow or maximum response time: $\max_j (C_j - r_j)$
  No starvation. Favor long jobs. Worst-case optimization.
- Maximum weighted flow: $\max_j w_j (C_j - r_j)$
  Gives back some importance to short jobs.
  Particular case of the *stretch* or *slowdown*:
  w_j=1/running time of the job on empty platform.

Outline

FIFO competitiveness

Theorem

First come, first served is:

- *optimal for the online minimization of max-flow*
- Δ*-competitive for the online minimization of sum-flow*
- Δ*-competitive for the online minimization of max-stretch*
- Δ^2*-competitive for the online minimization of sum-stretch*

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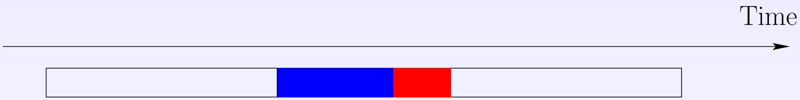
FIFO is optimal for max-flow

Consider any instance and a schedule Θ s.t. there exists two jobs executed consecutively: J_i and J_j with $r_i < r_j$ and $C_i \geq C_j$



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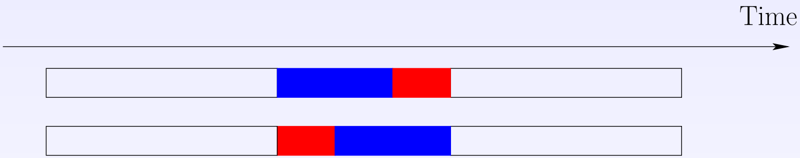


In schedule Θ' we exchange the execution order of J_i and J_j

$$\max_{1 \leq k \leq n} C'_k - r_k = \max\{ \max_{\substack{1 \leq k \leq n \\ k \notin \{i,j\}}} C_k - r_k, C'_i - r_i, C'_j - r_j \}$$

$$C'_i - r_i < C_i - r_i \quad \text{and} \quad C'_j - r_j = C_i - r_j < C_i - r_i$$

$$\Rightarrow \qquad \max_{1 \leq k \leq n} C'_k - r_k \leq \max_{1 \leq k \leq n} C_k - r_k$$

FIFO competitiveness for max-stretch

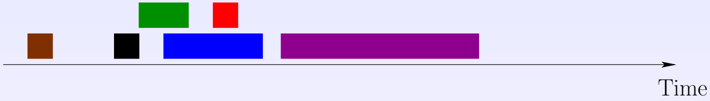
Theorem

FIFO is Δ competitive for maximum stretch minimization

This means that

1. FIFO has a competitive factor of Δ (i.e., on no instance is FIFO's max-stretch more than Δ that of the optimal solution)
2. This bound is tight (=cannot be improved)

Upper bound for max-stretch



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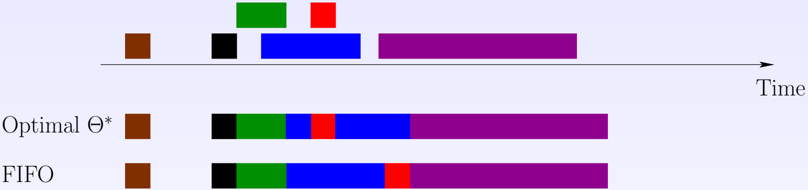

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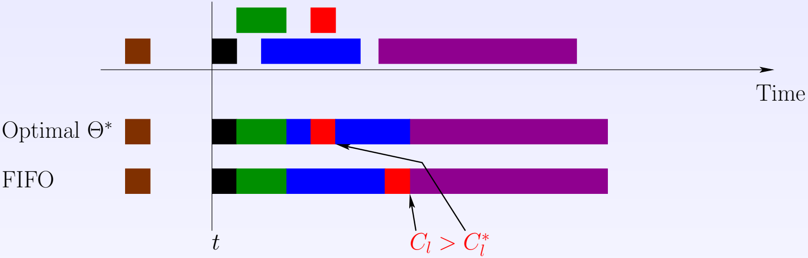
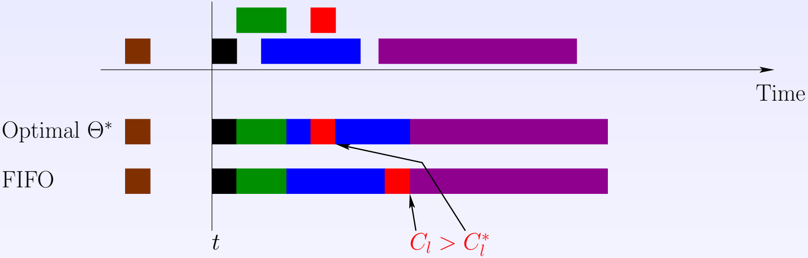


Any job J_l s.t. $\mathcal{S}_l > \mathcal{S}_l^*$ ($\Leftrightarrow C_l > C_l^*$)

t last time before C_l s.t. the processor was idle under FIFO.

t is the release date r_i of some job J_i.

Upper bound for max-stretch



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During $[r_i, C_l]$, FIFO exactly executes J_i, J_{i+1}, ..., J_{l-1}, J_l.

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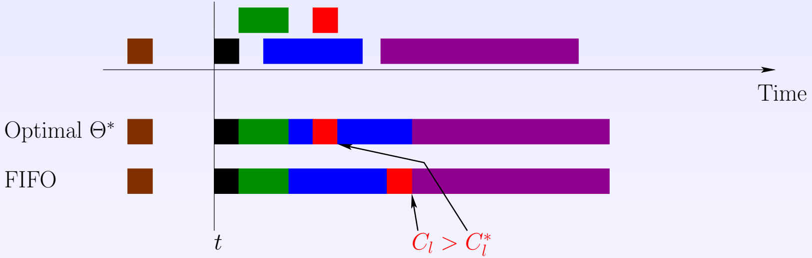
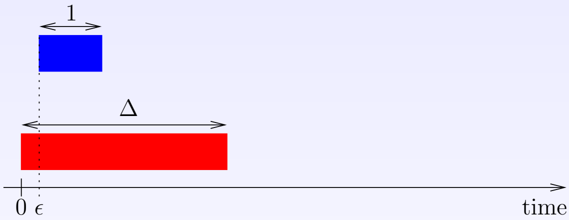


Any job J_l s.t. $\mathcal{S}_l > \mathcal{S}_l^*$ ($\Leftrightarrow C_l > C_l^*$)
During $[r_i, C_l]$, FIFO exactly executes J_i, J_{i+1}, ..., J_{l-1}, J_l.
As $C_l^* < C_l$, there is a job J_k, $i \leq k \leq l-1$ s.t. $C_k^* \geq C_l$. Then:

$$\mathcal{S}^* = \max_j \mathcal{S}_j^* \geq \mathcal{S}_k^* = \frac{C_k^* - r_k}{p_k} \geq \frac{C_l - r_l}{p_k} = \frac{C_l - r_l}{p_l} \frac{p_l}{p_k} \geq \mathcal{S}_l \times \frac{1}{\Delta}$$

$$\forall l, \mathcal{S}_l > \mathcal{S}_l^* \quad \Rightarrow \quad \Delta \times \mathcal{S}^* \geq \mathcal{S}_l$$

The bound is tight



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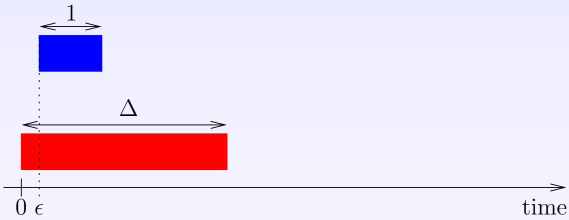

FIFO $\quad$ Max-stretch $= \frac{1+\Delta-\epsilon}{1}$

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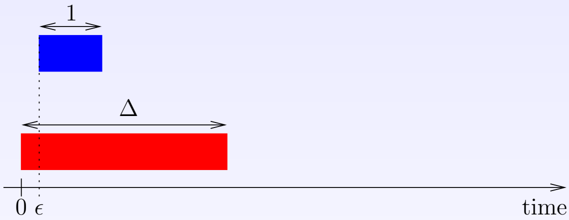

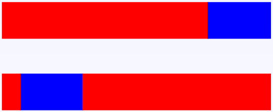

FIFO Max-stretch $= \frac{1+\Delta-\epsilon}{1}$

Optimal Max-stretch $= \frac{1+\Delta}{\Delta}$

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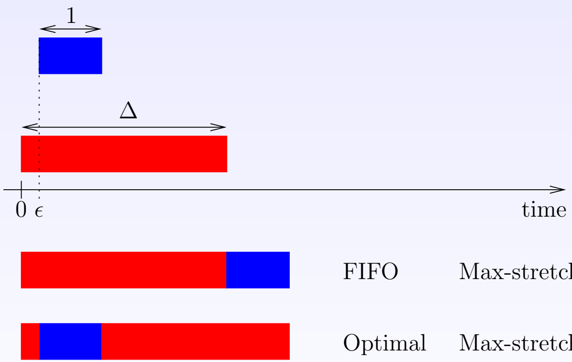

FIFO $\quad$ Max-stretch $= \frac{1+\Delta-\epsilon}{1}$

Optimal $\quad$ Max-stretch $= \frac{1+\Delta}{\Delta}$

Competitive ratio: $\frac{1+\Delta-\epsilon}{\frac{1+\Delta}{\Delta}} = \Delta\frac{1+\Delta-\epsilon}{1+\Delta} = \Delta - \epsilon\,\frac{\Delta}{1+\Delta} \geq \Delta - \epsilon$

Outline

Bound on the competitive ratio

Theorem

On one processor, any online scheduling algorithm with preemption minimizing the max-stretch has a competitive ratio greater than $\frac{1}{2}\Delta^{\sqrt{2}-1}$*, if the system receives at least jobs of three different sizes, and if* Δ *is the ratio between the size of the largest and the smallest job.*

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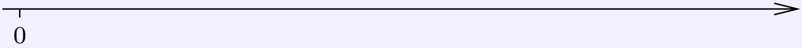
Theorem

On one processor, any online scheduling algorithm with preemption minimizing the max-stretch has a competitive ratio greater than $\frac{1}{2}\Delta^{\sqrt{2}-1}$, if the system receives at least jobs of three different sizes, and if Δ is the ratio between the size of the largest and the smallest job.

Proof principle: by contradiction we assume that there exists an algorithm and we build a sequence of jobs and a scenario to make the algorithm fail.

The adversary



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Achievable stretch: $\dfrac{2\delta - 0}{\delta} = 2.$

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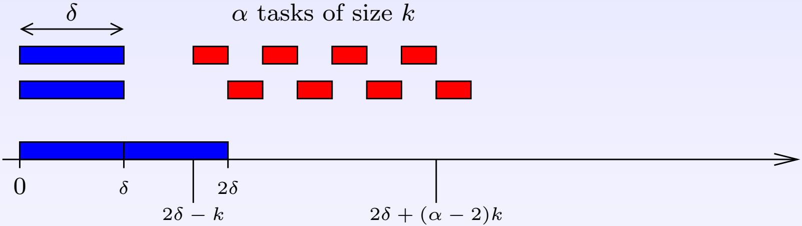
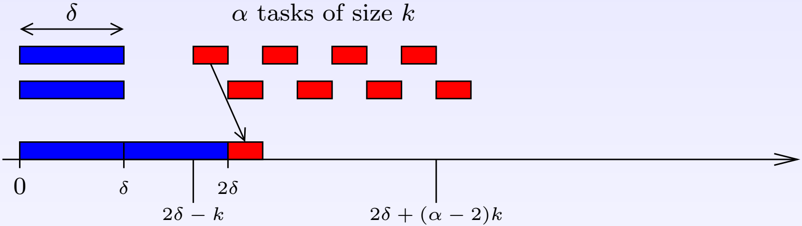


The job T_{2+j} arrives at time $2\delta + (j-2)k$.

The adversary



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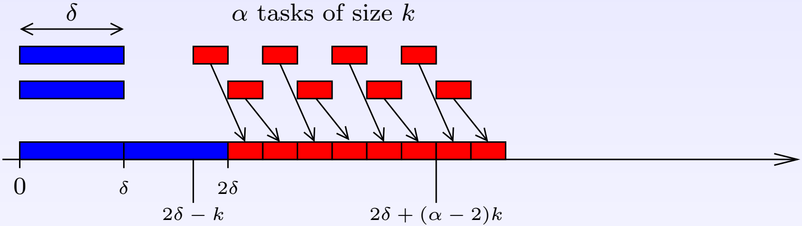
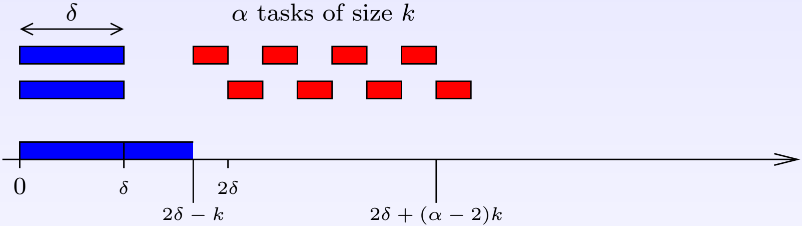


The job T_{2+j} arrives at time $2\delta + (j-2)k$.

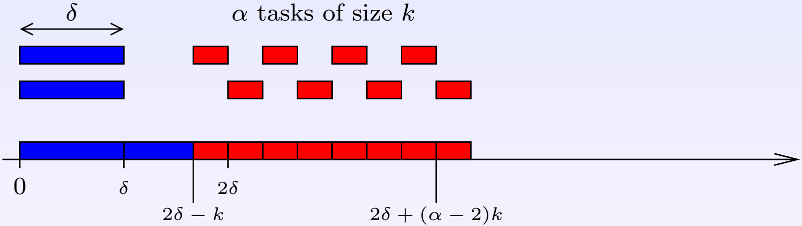
Achievable stretch: $\dfrac{(2\delta + jk) - (2\delta + (j-2)k)}{k} = 2.$

The adversary



In practice: we do not know what happens after $2\delta - k$.

The adversary



We want to forbid this case (each size-k job being executed at its release date).

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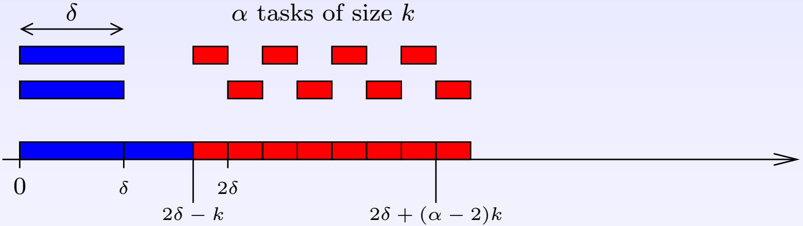

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The algorithm being $\frac{1}{2}\Delta^{\sqrt{2}-1}$-competitive, T_1 and T_2 must be completed at the latest at time: $2 \cdot \dfrac{1}{2}\Delta^{\sqrt{2}-1} \cdot \delta = 2 \cdot \dfrac{1}{2}\left(\dfrac{\delta}{k}\right)^{\sqrt{2}-1} \cdot \delta$

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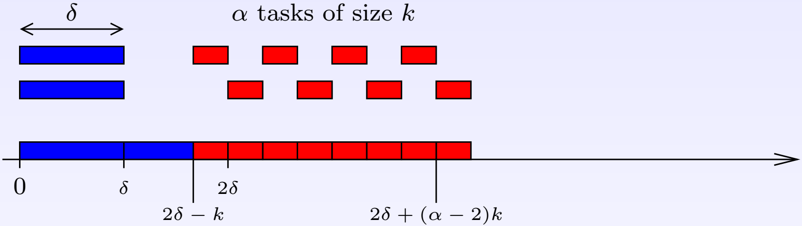

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We let $\alpha = \lceil 1 + k - \frac{2\delta}{k} \rceil$ and then $2\delta + (\alpha - 1)k \geq 2 \cdot \frac{1}{2}\left(\frac{\delta}{k}\right)^{\sqrt{2}-1} \cdot \delta$.

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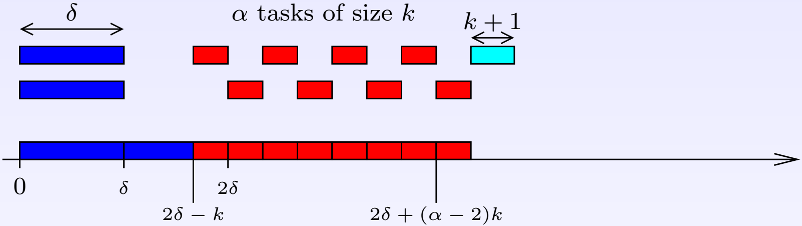

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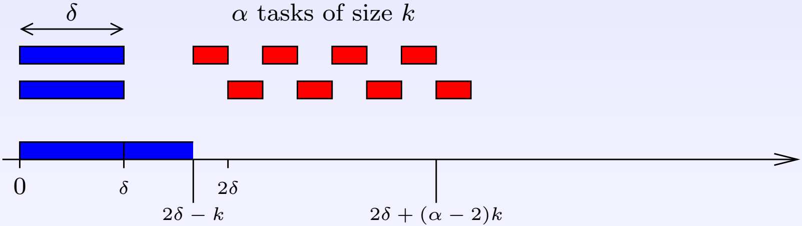

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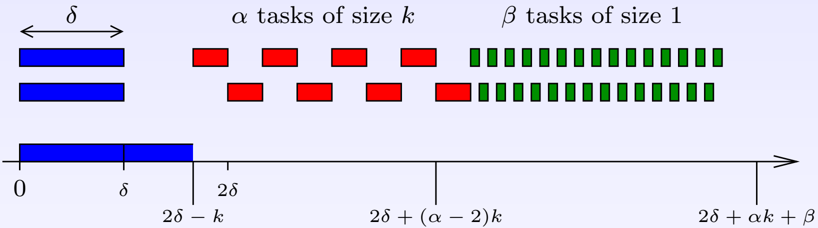
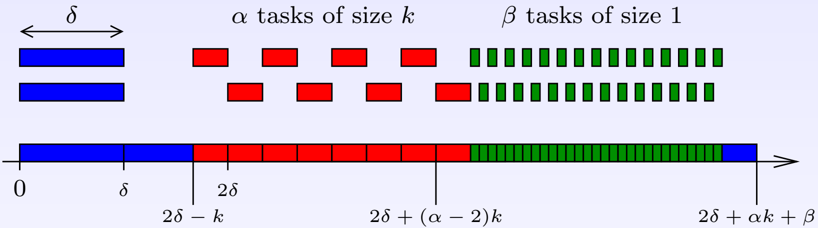


The job $T_{2+\alpha+j}$ arrives at time $2\delta + (\alpha - 1)k + (j - 1)$.

The adversary



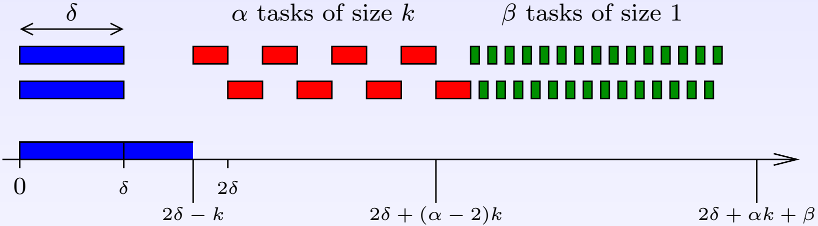
Achievable stretch (off-line)

Stretch of each job of size k or 1 : 1.

Stretch of T_1 or T_2: $\dfrac{2\delta + \alpha k + \beta}{\delta}$

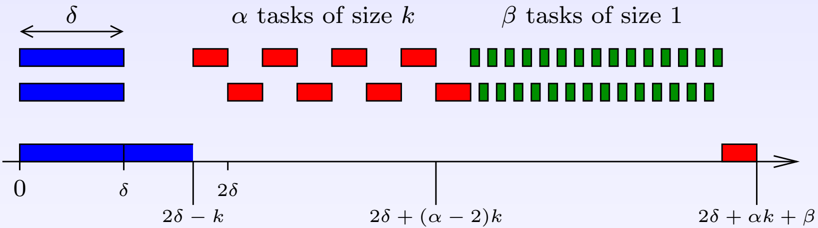
Optimal stretch $\leq \dfrac{2\delta + \alpha k + \beta}{\delta}$

The adversary



Achievable stretch (online)

The adversary

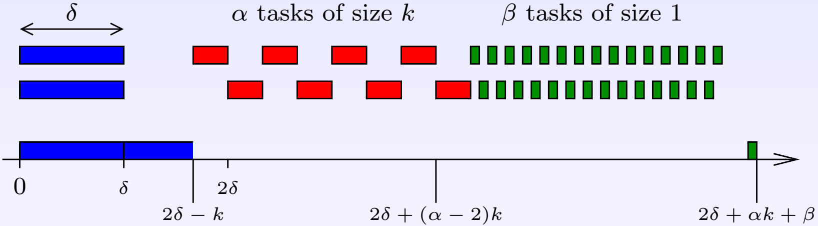


Achievable stretch (online)

The last completed job is of size k.

$$\text{Stretch} \geq \frac{(2\delta + \alpha k + \beta) - (2\delta + (\alpha - 2)k)}{k} = 2 + \frac{\beta}{k}.$$

The adversary

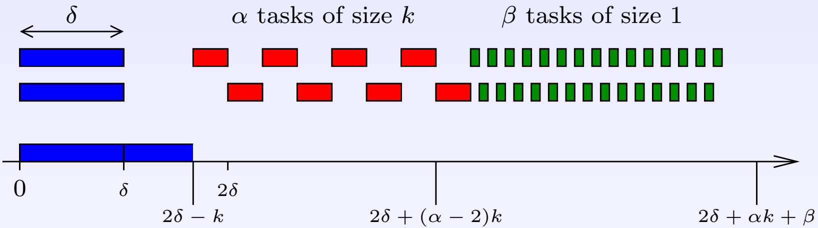


Achievable stretch (online)

The last completed job is of size 1.

$$\text{Stretch} \geq \frac{(2\delta + \alpha k + \beta) - (2\delta + (\alpha - 1)k + (\beta - 1))}{1} = k + 1.$$

The adversary



Achievable stretch (online)

$$\text{Stretch} \geq \min\left\{2 + \frac{\beta}{k}, k+1\right\}$$

We let: $\beta = \lceil k(k-1) \rceil$

Then: stretch $\geq k + 1$.

The adversary: summing things up

$$\alpha = \left\lceil 1 + k - \frac{2\delta}{k} \right\rceil$$

$$\beta = \lceil k(k-1) \rceil$$

Optimal stretch $\leq \dfrac{2\delta + \alpha k + \beta}{\delta}$

Achieved stretch $\geq k + 1$.

The adversary: summing things up

$$\alpha = \left\lceil 1 + k - \frac{2\delta}{k} \right\rceil$$

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Optimal stretch $\leq \dfrac{2\delta + \alpha k + \beta}{\delta}$

Achieved stretch $\geq k + 1$.

We let $k = \delta^{2-\sqrt{2}}$

The adversary: summing things up

$$\alpha = \left\lceil 1 + k - \frac{2\delta}{k} \right\rceil$$

$$\beta = \lceil k(k-1) \rceil$$

Optimal stretch $\leq \dfrac{2\delta + \alpha k + \beta}{\delta}$

Achieved stretch $\geq k + 1$.

We let $k = \delta^{2-\sqrt{2}}$

Therefore $k + 1 > \left(\frac{1}{2}\delta^{\sqrt{2}-1}\right)\left(\frac{2\delta + \alpha k + \beta}{\delta}\right)$

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Lower bound as a function of n

Theorem

There is no c-competitive preemptive online algorithm minimizing the maximum stretch with $c < n$

Principle of the proof

- We suppose there exists an algorithm whose ratio $c = n - \epsilon$
- n jobs are released at time 0
- Whatever the scheduler does, no job completes before time n
- Jobs are sorted by non-decreasing cumulative computation time computed at time n: the $(i+1)$-th job is of size λ^{i-1}
- The maximum stretch is at least n (first job has size 1 and is not completed at n)
- Optimal: execute jobs in Shortest Processing Time first order:

$$\frac{\sum_{j=1}^{i} \lambda^{j-1}}{\lambda^{i-1}} = \frac{\lambda^i - 1}{\lambda^{i-1}(\lambda - 1)} \xrightarrow[\lambda \to +\infty]{} 1$$

EquiPartition

Theorem

EquiPartition is n-competitive for the minimization of maximum stretch.

However, EquiPartition is at best $\frac{\Delta+1}{2+\ln(\Delta)}$ competitive (when FIFO is Δ competitive)

Outline

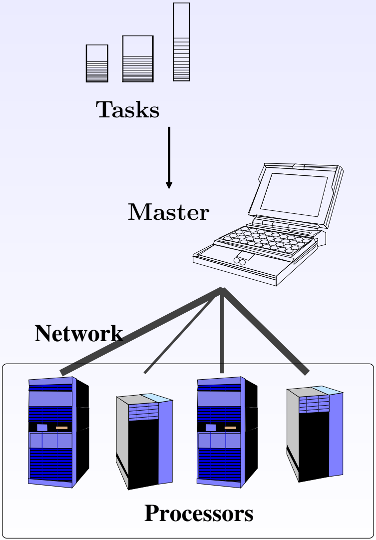
The scheduling problem

The scheduler

- Gather the tasks
- Send them to the processors

The aim

Distribute *identical* tasks to the processors, in order to process these tasks

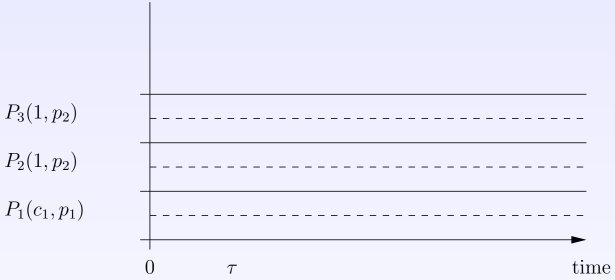


The scheduling problem

Formally

- n tasks, m processors
- p_j: processing time of a task on processor j
- c_j: time to send a task from the master to the worker j
- r_i: arrival date
- C_i: completion time
- The objective function:
  - maximal flow: $\max C_i - r_i$

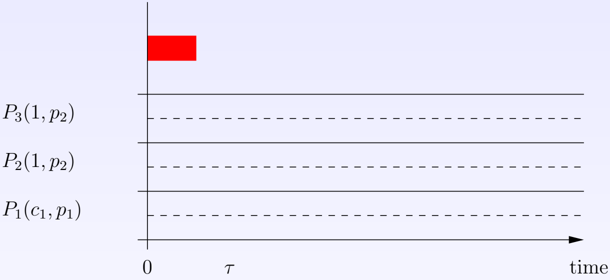
Finding a lower bound on the competitiveness (1)



Idea:

- a fast processor with slow communications ($c_1 > 1$)
- two identical and slow processors, with fast communications
- if only one task, one must choose the fast processor ($c_1 + p_1 < 1 + p_2$)

Finding a lower bound on the competitiveness (1)



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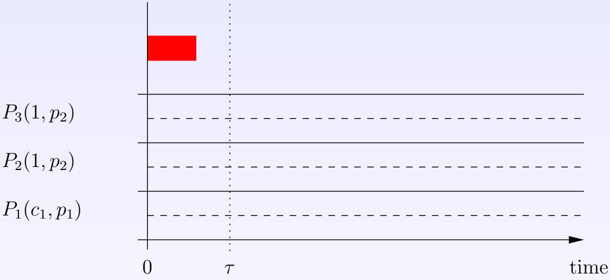
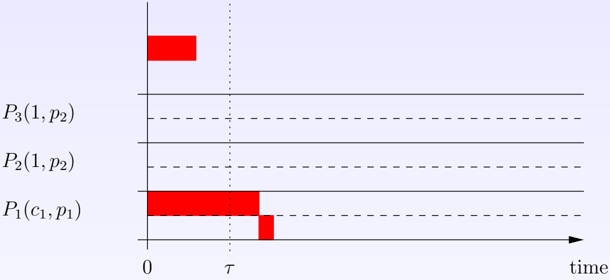


We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

Finding a lower bound on the competitiveness (1)



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1. Optimal : task on P_1, max-flow $\geq c_1 + p_1$.

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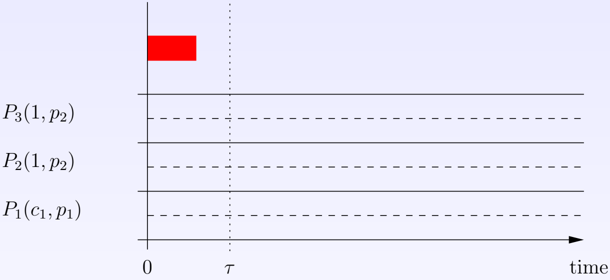
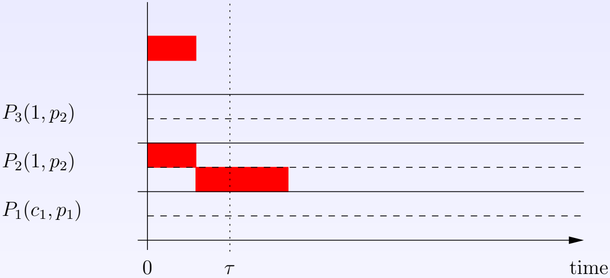


We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

1. Optimal : task on P_1, max-flow $\geq c_1 + p_1$.
2. Nothing done: max-flow $\geq \tau + c_1 + p_1$, ratio $\geq \frac{\tau + c_1 + p_1}{c_1 + p_1}$.

Finding a lower bound on the competitiveness (1)

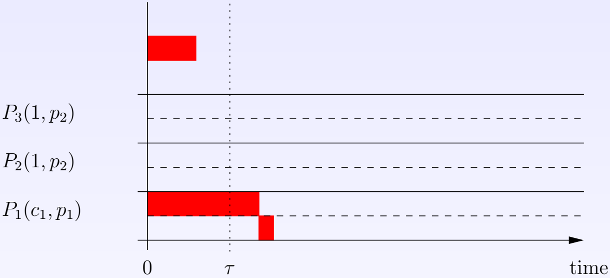


We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

1. Optimal : task on P_1, max-flow $\geq c_1 + p_1$.
2. Nothing done: max-flow $\geq \tau + c_1 + p_1$, ratio $\geq \frac{\tau+c_1+p_1}{c_1+p_1}$.
3. Task send to P_2, max-flow $\geq 1 + p_2$. Ratio $\geq \frac{1+p_2}{c_1+p_1}$.

We want to force the algorithm to process the first task on P_1.

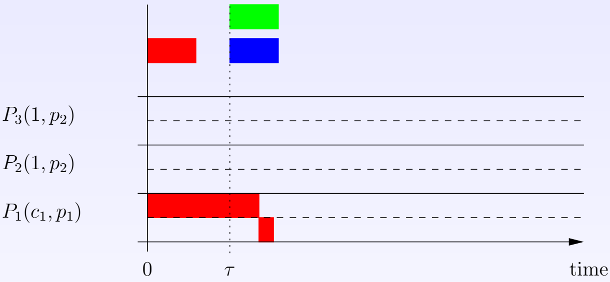
Finding a lower bound on the competitiveness (1)



We look at time $\tau \geq 1$ to see what has happened. We will choose τ, c_1, p_1 and p_2 such that:

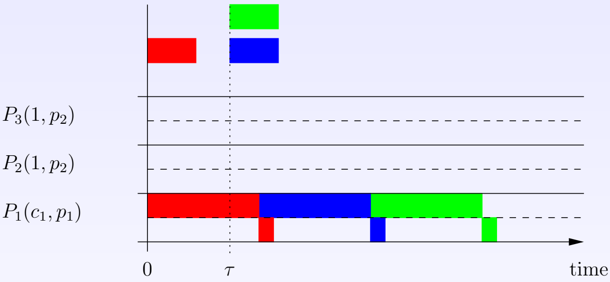
$$\min\left\{\frac{1+p_2}{c_1+p_1}, \frac{\tau+c_1+p_1}{c_1+p_1}\right\} \geq \rho$$

Finding a lower bound on the competitiveness (1)



At time τ we send two new tasks.
We consider all the possible cases.

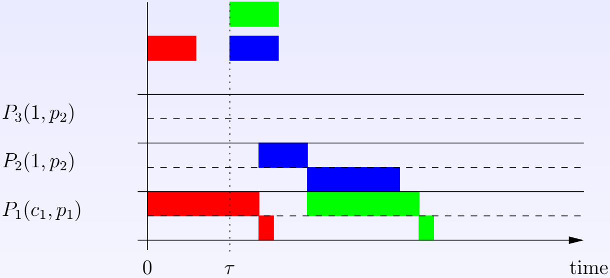
Finding a lower bound on the competitiveness (1)



At time τ we send two new tasks.
The two tasks are executed on P_1:

$$\max\{c_1 + p_1,$$
$$\max\{\max\{c_1, \tau\} + c_1 + p_1, c_1 + 2p_1\} - \tau,$$
$$\max\{\max\{c_1, \tau\} + c_1 + p_1 + \max\{c_1, p_1\}, c_1 + 3p_1\} - \tau\}$$

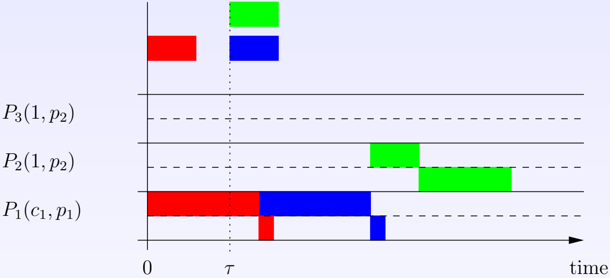
Finding a lower bound on the competitiveness (1)



At time τ we send two new tasks.
The first of the two tasks is executed on P_2 (or P_3), and the other one on P_1.

$$\max\{c_1 + p_1,$$
$$(\max\{c_1, \tau\} + c_2 + p_2) - \tau,$$
$$\max\{\max\{c_1, \tau\} + c_2 + c_1 + p_1, c_1 + 2p_1\} - \tau\}$$

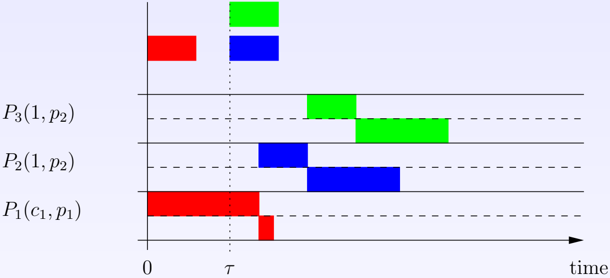
Finding a lower bound on the competitiveness (1)



At time τ we send two new tasks.
The first of the two tasks is executed on P_1, and the other one on P_2 (or P_3).

$$\max\{c_1 + p_1, \\ \max\{\max\{c_1, \tau\} + c_1 + p_1, c_1 + 2p_1\} - \tau, \\ (\max\{c_1, \tau\} + c_1 + c_2 + p_2) - \tau\}$$

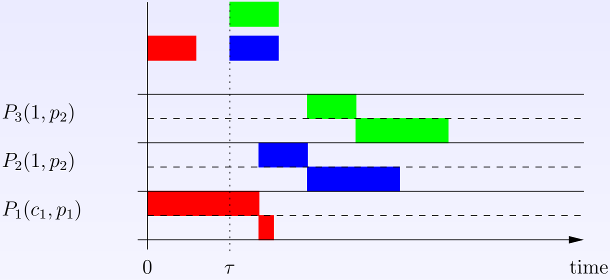
Finding a lower bound on the competitiveness (1)



At time τ we send two new tasks.
One of the two tasks is executed on P_2 and the other one on P_3.

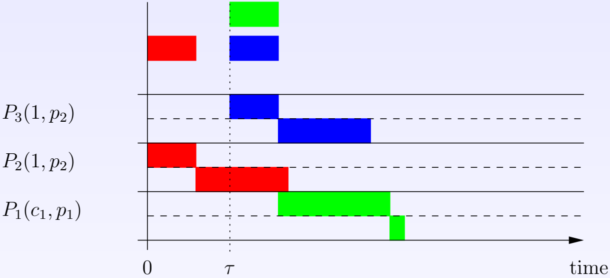
$$\max\{c_1+p_1, (\max\{c_1,\tau\}+c_2+p_2)-\tau, (\max\{c_1,\tau\}+c_2+c_2+p_2)-\tau\}$$

Finding a lower bound on the competitiveness (1)



At time τ we send two new tasks.
The case where both tasks are executed on P_2 (or both on P_3) is worse than the previous one, therefore, we do not need to study it.

Finding a lower bound on the competitiveness (1)



At time τ we send two new tasks.
The (desired) optimal: the first task on P_2, the second on P_3, and the third on P_1.

$$\max\{c_2+p_2, (\max\{c_2, \tau\}+c_2+p_2)-\tau, (\max\{c_2, \tau\}+c_2+c_1+p_1)-\tau\}$$

Finding a lower bound on the competitiveness (2)

Lower bound on the competitiveness of any online algorithm:

$$\min\begin{cases}\frac{\tau+c_1+p_1}{c_1+p_1},\\[2ex]\frac{1+p_2}{c_1+p_1},\\[2ex]\dfrac{\min\begin{cases}\max\{c_1+p_1,\max\{\max\{c_1,\tau\}+c_1+p_1,c_1+2p_1\}-\tau,\\ \qquad\max\{\max\{c_1,\tau\}+c_1+p_1+\max\{c_1,p_1\},c_1+3p_1\}-\tau\}\\ \max\{c_1+p_1,(\max\{c_1,\tau\}+c_2+p_2)-\tau,\max\{\max\{c_1,\tau\}+c_2+c_1+p_1,c_1+2p_1\}-\tau\}\\ \max\{c_1+p_1,\max\{\max\{c_1,\tau\}+c_1+p_1,c_1+2p_1\}-\tau,(\max\{c_1,\tau\}+c_1+c_2+p_2)-\tau\}\\ \max\{c_1+p_1,(\max\{c_1,\tau\}+c_2+p_2)-\tau,(\max\{c_1,\tau\}+c_2+c_2+p_2)-\tau\}\end{cases}}{\max\{c_2+p_2,(\max\{c_2,\tau\}+c_2+p_2)-\tau,(\max\{c_2,\tau\}+c_2+c_1+p_1)-\tau\}}\end{cases}$$

Problem : to find τ, c_1, p_1, and p_2 (as $c_2 = 1$) which maximizes this lower bound.
Constraints : $c_1 + p_1 < 1 + p_2$.

Finding a lower bound on the competitiveness (3)

1. Numeric resolution

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3. New system:

$$\min \begin{cases} \frac{\tau + c_1}{c_1} \\ \frac{1+p_2}{c_1} \\ \frac{\min \begin{cases} 3c_1 - \tau \\ c_1 + 1 - \tau + p_2 \\ 2c_1 - \tau + 1 + p_2 \\ c_1 + 2 + p_2 - \tau \end{cases}}{1+p_2} \end{cases} = \min \begin{cases} \frac{\tau + c_1}{c_1} \\ \frac{1+p_2}{c_1} \\ \frac{c_1 + 1 - \tau + p_2}{1+p_2} \end{cases}$$

Finding a lower bound on the competitiveness (3)

1. Numeric resolution
2. Characterization of the shape of the optimal : $\tau < c_1$, $p_1 = 0$, etc.
3. New system:

$$\min \begin{cases} \frac{\tau+c_1}{c_1} \\ \frac{1+p_2}{c_1} \\ \frac{\min \begin{cases} 3c_1 - \tau \\ c_1 + 1 - \tau + p_2 \\ 2c_1 - \tau + 1 + p_2 \\ c_1 + 2 + p_2 - \tau \end{cases}}{1+p_2} \end{cases} = \min \begin{cases} \frac{\tau+c_1}{c_1} \\ \frac{1+p_2}{c_1} \\ \frac{c_1+1-\tau+p_2}{1+p_2} \end{cases}$$

4. Solution: $c_1 = 2(1+\sqrt{2})$, $p_2 = \sqrt{2}c_1 - 1$, $\tau = 2$, $\rho = \sqrt{2}$.