Steady-State Scheduling

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Overview

1. The context
2. Routing packets with fixed communication routes
3. Resolution of the "fluidified" problem
4. Building a schedule
5. Packet routing without fixed path
6. Bags of sequential applications

Overview

Platform

Platform: heterogeneous and distributed:

- processors with different capabilities;
- communication links of different characteristics.

Applications

Application made of a very (very) large number of tasks, the tasks can be clustered into a finite number of types, all tasks of a same type having the same characteristics.

Bag-of-tasks applications, parameter sweep applications, etc.

Principle

When we have a very large number of identical tasks to execute, we can imagine that, after some initiation phase, we will reach a (long) steady-state, before a termination phase.

If the steady-state is long enough, the initiation and termination phases will be negligible.

Overview

The problem

Problem: sending a set of message flows.

In a communication network, several flow of packets must be dispatched, each packet flow must be sent from a source to a destination, while following a given path linking the source to the destination.

Notations

- (V, A) a directed graph, representing the communication network.
- A set of n_c flows which must be dispatched.
- The k-th flow is denoted (s_k, t_k, P_k, n_k), where
  - s_k is the source of packets;
  - t_k is the destination;
  - P_k is the path to be followed;
    We denote by $a_{k,i}$ the i-th edge in the path P_k.
  - n_k is the number of packets in the flow.

Hypotheses

- A packet goes through an edge A in a unit of time.
- At a given time, a single packet traverses a given edge.

Objective

We must decide which packet must go through a given edge at a given time, in order to minimize the overall execution time.

Lower bound on the duration of schedules

We call **congestion** of edge $a \in A$, and we denote by C_a, the total number of packets which go through edge a:

$$C_a = \sum_{k \mid a \in P_k} n_k \qquad C_{\max} = \max_a C_a$$

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A "fluid" (fractional) resolution of our problem will give us a solution which executes in a time $C_{\max}$.

Overview

Fluidified (fractional) version: notations

Principle:

- we do not look for an integral solution but for a rational one.

- $n_{k,i}(t)$ (fractional) number of packets waiting at the entrance of the i-th edge of the k-th path, at time t.

- $T_{k,i}(t)$ is the overall time used by the edge $a_{k,i}$ for packets of the k-th flow, during the interval of time $[0;t]$.

Fluidified (fractional) version: writing the equations

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3. Resource constraints

$$\sum_{(k,i) \ | \ a_{k,i}=a} T_{k,i}(t_2) - T_{k,i}(t_1) \leq t_2 - t_1, \forall a \in A, \forall t_2 \geq t_1 \geq 0$$

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4. Objective

$$\text{MINIMIZE } C_{\text{frac}} = \int_0^\infty \mathbb{1}\left(\sum_{k,i} n_{k,i}(t)\right) dt$$

Lower bound

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- For each edge a:

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As long as $t < C_a$, there are packets in the system.

Therefore, $C_{\text{frac}} \geq \max_a C_a = C_{\max}$

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This solution is a schedule of makespan $C_{\max}$. We still have to show that it is feasible.

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   $$\frac{C_a}{C_{\max}}(t_2 - t_1) \leq t_2 - t_1$$

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- Period of the schedule: $\Omega + D_{\max}$.

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(If less than m_k packets are waiting in the entrance of a at time $j(\Omega + D_{\max})$, a forwards what is available and remains idle longer.)

Feasibility of the schedule

$$\begin{aligned}
\sum_{(k,i)|a_{k,i}=a} m_k &= \sum_{(k,i)|a_{k,i}=a} \left\lceil \frac{n_k \Omega}{C_{\max}} \right\rceil \\
&\leq \sum_{(k,i)|a_{k,i}=a} \left(\frac{n_k \Omega}{C_{\max}} + 1 \right) \\
&\leq \frac{C_a}{C_{\max}} \Omega + D_a \\
&\leq \Omega + D_{\max}
\end{aligned}$$

Behavior of the sources

- $N_{k,i}(t)$: number of packets of the k-th flow waiting at the entrance of the i-th edge, at time t.
- $a_{k,1}$ sends m_k packets during $[0, \Omega + D_{\max}]$.
  $N_{k,1}(\Omega + D_{\max}) = n_k - m_k$
- $a_{k,1}$ sends m_k packets during $[\Omega + D_{\max}, 2(\Omega + D_{\max})]$.
  $N_{k,1}(2(\Omega + D_{\max})) = n_k - 2m_k$
- We let $T = \left\lceil \frac{C_{\max}}{\Omega} \right\rceil (\Omega + D_{\max})$

$$N_{k,1}(T) \leq n_k - \frac{T}{\Omega + D_{\max}} m_k \leq n_k - \frac{n_k \Omega}{C_{\max}} \frac{C_{\max}}{\Omega} = 0$$

Propagation delay

- $a_{k,1}$ sends m_k packets during $[0, \Omega + D_{\max}]$.
  $N_{k,1}(\Omega + D_{\max}) = n_k - m_k$ $\quad$ $N_{k,2}(\Omega + D_{\max}) = m_k$
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  $N_{k,3}(2(\Omega + D_{\max})) = m_k$ $\quad$ $N_{k,i\geq 4}(2(\Omega + D_{\max})) = 0$

- The delay between the time a packet traverses the first edge of the path P_k and the time it traverses its last edge is, at worst:
  $$(|P_k| - 1)(\Omega + D_{\max})$$
  We let $L = \max_k |P_k|$.

Makespan of the schedule

$$
\begin{aligned}
C_{\mathsf{total}} &\le T + (L-1)(\Omega + D_{\max}) \\
&= \left\lceil \frac{C_{\max}}{\Omega} \right\rceil (\Omega + D_{\max}) + (L-1)(\Omega + D_{\max}) \\
&\le \left(\frac{C_{\max}}{\Omega} + 1 \right) (\Omega + D_{\max}) + (L-1)(\Omega + D_{\max}) \\
&= C_{\max} + L D_{\max} + \frac{D_{\max} C_{\max}}{\Omega} + L\Omega
\end{aligned}
$$

The upper bound is minimized by $\Omega = \sqrt{\dfrac{D_{\max} C_{\max}}{L}}$

$$
C_{\mathsf{total}} \le C_{\max} + 2\sqrt{C_{\max} D_{\max} L} + D_{\max} L
$$

Asymptotic optimality

$$C_{\max} \leq C^* \leq C_{\textsf{total}} \leq C_{\max} + 2\sqrt{C_{\max} D_{\max} L} + D_{\max} L$$

$$1 \leq \frac{C_{\textsf{total}}}{C_{\max}} \leq 1 + 2\sqrt{\frac{D_{\max} L}{C_{\max}}} + \frac{D_{\max} L}{C_{\max}}$$

$$\text{With } \Omega = \sqrt{\frac{D_{\max} C_{\max}}{L}}$$

Resources needed

$$\sum_{(k,i)|a_{k,i}=a} m_k \leq \sum_{(k,i)|a_{k,i}=a} \left(\frac{n_k}{C_{\max}} \sqrt{\frac{D_{\max} C_{\max}}{L}} + 1 \right)$$

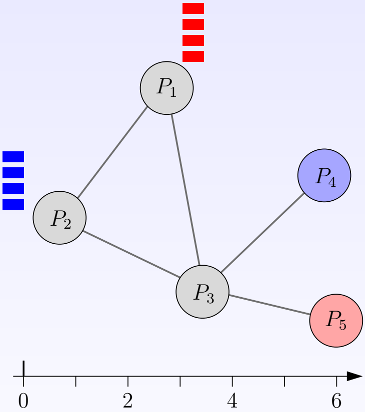
$$\leq \sqrt{\frac{D_{\max} C_{\max}}{L}} + D_{\max}$$

Conclusion

- We forget the initiation and termination phases
- Rational resolution of the steady-state
- Rounds whose size is the square-root of the solution:
  - Each round “loses” a constant amount of time
  - The sum of the waisted times increases less quickly than the schedule
  - Buffers of size the square-root of the solution

Overview

Packet routing without fixed path



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- packets of a same collection may follow different paths
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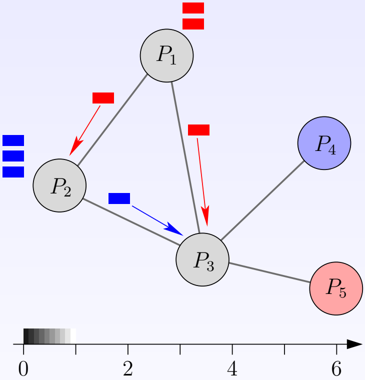

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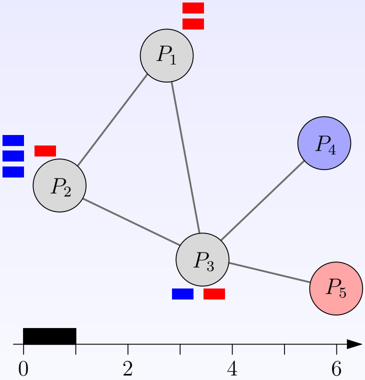

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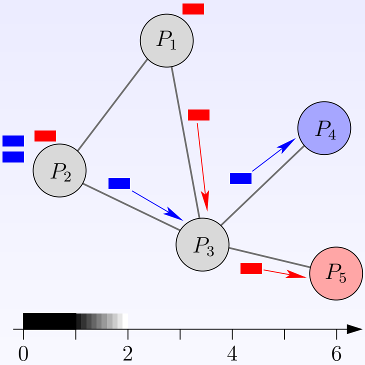

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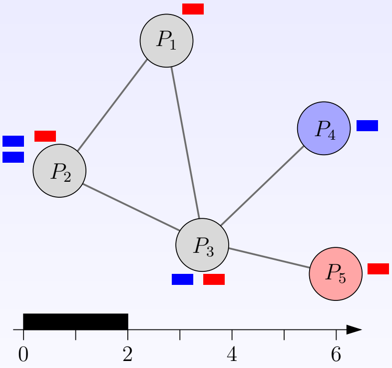

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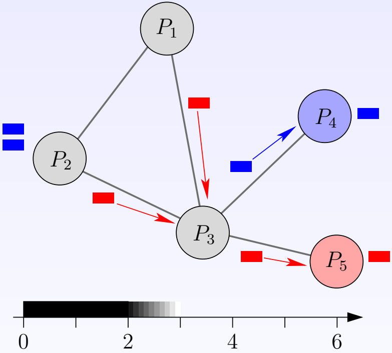

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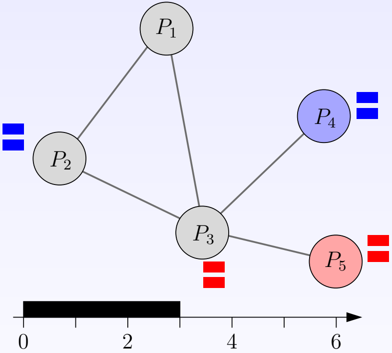

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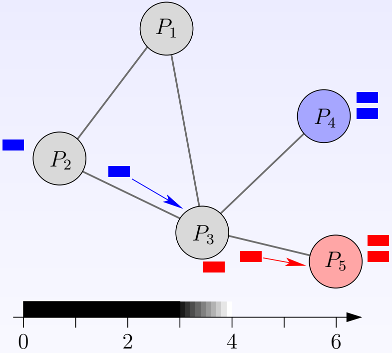

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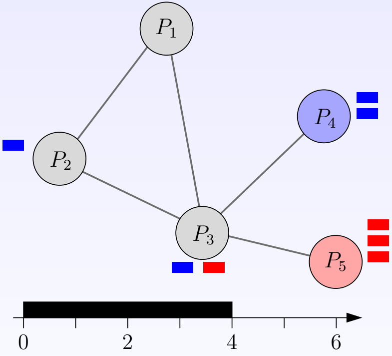

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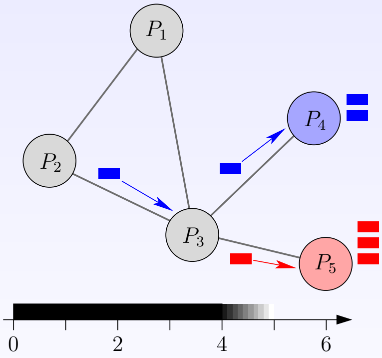

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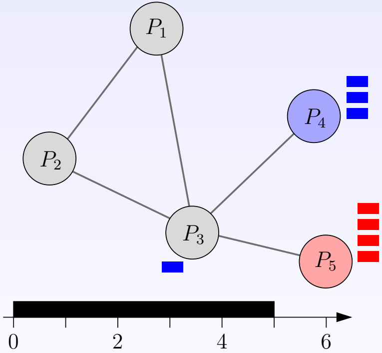

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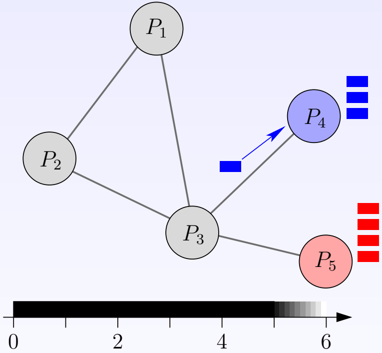

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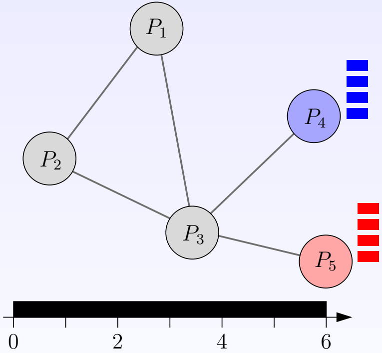

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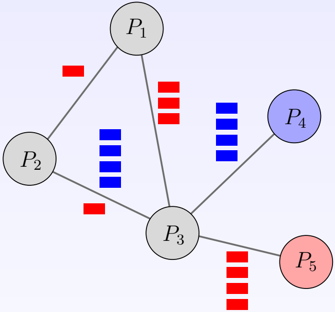

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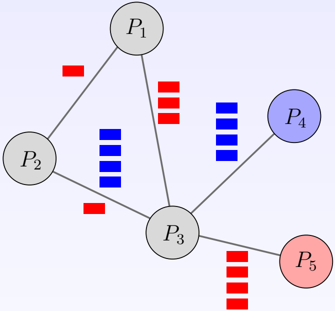
- $n_{i,j}^{k,l}$: total number of packets routed from k to l and crossing edge (i, j)

Packet routing without fixed path



- n_c collections of packets to be routed
- packets of a same collection may follow different paths
- $n^{k,l}$: total number of packets to be routed from k to l
- rule: one edge cannot carry two packets at the same time

- $n_{i,j}^{k,l}$: total number of packets routed from k to l and crossing edge (i,j)
- Congestion: $C_{i,j} = \sum_{(k,l)|n^{k,l}>0} n_{i,j}^{k,l}; \quad C_{\max} = \max_{i,j} C_{i,j}$

Equations (1/2)

1. Initialization

$$\sum_{j|(k,j)\in A} n_{k,j}^{k,l} = n^{k,l}$$

2. Reception

$$\sum_{i|(i,l)\in A} n_{i,l}^{k,l} = n^{k,l}$$

3. Conservation law

$$\sum_{i|(i,j)\in A} n_{i,j}^{k,l} = \sum_{i|(j,i)\in A} n_{j,i}^{k,l} \quad \forall (k,l), j \neq k, j \neq l$$

Equations (2/2)

4. Congestion

$$C_{i,j} = \sum_{(k,l)|n^{k,l}>0} n_{i,j}^{k,l}$$

5. Objective function

$$C_{\max} \geq C_{i,j}, \qquad \forall i, j$$

Minimize $C_{\max}$

Linear program in rational numbers: polynomial-time solution.

Solution:
number of messages $n_{i,j}^{k,l}$ on each edge to minimize congestion

Routing algorithm

1. Computing optimal solution $C_{\max}$ of previous linear program
2. Consider periods of length Ω (to be defined later)
3. During each time-interval $[p\Omega, (p+1)\Omega]$, follow the optimal solution: edge (i,j) forwards:

   $$m_{i,j}^{k,l} = \left\lfloor \frac{n_{i,j}^{k,l}\Omega}{C_{\max}} \right\rfloor$$

   packets that go from k to l.
   (if available)

4. number of such periods: $\left\lceil \frac{C_{\max}}{\Omega} \right\rceil$
5. After time-step

   $$T \equiv \left\lceil \frac{C_{\max}}{\Omega} \right\rceil \Omega \leq C_{\max} + \Omega$$

   sequentially process M residual packets; this takes no longer than ML time-steps, where L is the maximum length of a simple path in the network

Feasibility

$$\sum_{(k,l)} m_{i,j}^{k,l} \leq \sum_{(k,l)} \frac{n_{i,j}^{k,l}\Omega}{C_{\max}} = \frac{C_{i,j}\Omega}{C_{\max}} \leq \Omega$$

Makespan

- Define Ω as $\Omega = \sqrt{C_{\max} n_c}$.

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- Total number of packets still inside network at time-step T is at most
$$2|A|\sqrt{C_{\max} n_c} + |A| n_c$$

Makespan

- Define Ω as $\Omega = \sqrt{C_{\max} n_c}$.
- Total number of packets still inside network at time-step T is at most

$$2|A|\sqrt{C_{\max} n_c} + |A|n_c$$

- Makespan:

$$C_{\max} \leq C^* \leq C_{\max} + \sqrt{C_{\max} n_c} + 2|A|\sqrt{C_{\max} n_c}|V| + |A|n_c|V|$$

$$C^* = C_{\max} + O(\sqrt{C_{\max}})$$

Steady-state scheduling

Background Approach pioneered by Bertsimas and Gamarnik

Rationale Maximize throughput (total load executed per period)

Simplicity Relaxation of makespan minimization problem

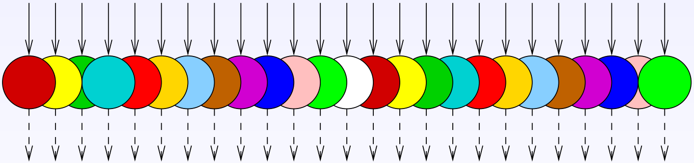
- Ignore initialization and clean-up phases
- Precise ordering/allocation of tasks/messages not needed
- Characterize resource activity during each time-unit:
  - which (rational) fraction of time is spent computing for which application?
  - which (rational) fraction of time is spent receiving or sending to which neighbor?

Efficiency Periodic schedule, described in compact form

Overview

Application graph

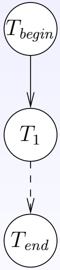
n problem instances $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \ldots, \mathcal{P}^{(n)}$, where n is large



Application graph

n problem instances $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \ldots, \mathcal{P}^{(n)}$, where n is large

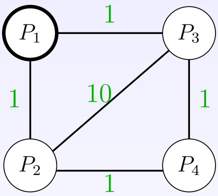
Each problem corresponds to a copy of the same task graph $G_A = (V_A, E_A)$, the application graph



T_{begin} et T_{end} are fictitious tasks, used to model the scattering of input files and the gathering of output files

Platform graph

Target platform represented by platform graph $G_P = (V_P, E_P)$

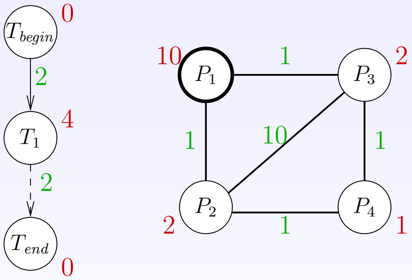


Edge $P_i \rightarrow P_j$ is labeled with $c_{i,j}$: time needed to send a unit-length message from P_i to P_j

Communication model: full overlap, one-port for incoming and outgoing messages

Computations and communications

P_i requires $w_{i,k}$ time-units to process task T_k ($k \in \{begin, 1, end\}$).



Edge $e_{k,l} : T_k \to T_l$ in G_A is labeled with $data_{k,l}$: data volume generated by T_k and used by T_l

Transfer time of a file $e_{k,l}$ from P_i to P_j: $data_{k,l} \times c_{i,j}$

Definitions

Allocation An allocation is a pair of mappings: $\pi : V_A \mapsto V_P$ and $\sigma : E_A \mapsto \{\text{paths in } G_P\}$

Schedule A schedule associated to an allocation (π, σ) is a pair of mappings: $t_\pi : V_A \mapsto \mathbb{R}$ and application $t_\sigma : E_A \times E_P \mapsto \mathbb{R}$, satisfying to:

- precedence constraints
- resource constraints on processors
- resource constraints on network links
- one-port constraints

Activity variables

$cons(P_i, T_k)$: average number of tasks of type T_k processed by P_i every time-unit

$$\forall P_i, \forall T_k \in V_A,\ 0 \leq cons(P_i, T_k) \times w_{i,k} \leq 1$$

$sent(P_i \rightarrow P_j, e_{k,l})$: average number of files of type $e_{k,l}$ sent from P_i to P_j every time-unit

$$\forall P_i, P_j,\ 0 \leq sent(P_i \rightarrow P_j, e_{k,l}) \times (data_{k,l} \times c_{i,j}) \leq 1$$

Steady-state equations

1. One-port for outgoing communications. P_i sends messages to its neighbors sequentially

$$\forall P_i, \sum_{P_i \to P_j} \sum_{e_{k,l} \in E_A} \left(sent(P_i \to P_j, e_{k,l}) \times data_{k,l} \times c_{i,j} \right) \leq 1$$

2. One-port for ingoing communications. P_i receives messages sequentially

$$\forall P_i, \sum_{P_j \to P_i} \sum_{e_{k,l} \in E_A} \left(sent(P_j \to P_i, e_{k,l}) \times data_{k,l} \times c_{j,i} \right) \leq 1$$

3. Overlap. Computations and communications take place simultaneously

$$\forall P_i, \sum_{T_k \in V_A} cons(P_i, T_k) \times w_{i,k} \leq 1$$

Conservation law

Consider a processor P_i and an edge $e_{k,l}$ of the application graph:

Files of type $e_{k,l}$ received: $\sum_{P_j \to P_i} sent(P_j \to P_i, e_{k,l})$

Files of type $e_{k,l}$ generated: $cons(P_i, T_k)$

Files of type $e_{k,l}$ consumed: $cons(P_i, T_l)$

Files of type $e_{k,l}$ sent: $\sum_{P_i \to P_j} sent(P_i \to P_j, e_{k,l})$

In steady state:

$$\forall P_i, \forall e_{k,l} : T_k \to T_l \in E_A,$$

$$\sum_{P_j \to P_i} sent(P_j \to P_i, e_{k,l}) + cons(P_i, T_k) =$$

$$\sum_{P_i \to P_j} sent(P_i \to P_j, e_{k,l}) + cons(P_i, T_l)$$

Upper bound for the throughput

$$\textsc{Maximize}\ \rho = \sum_{i=1}^{p} cons(P_i, T_{end}),$$

UNDER THE CONSTRAINTS

$$\begin{cases}
\text{(1a)} & \forall P_i, \forall T_k \in V_A,\ 0 \leq cons(P_i, T_k) \times w_{i,k} \leq 1 \\
\text{(1b)} & \forall P_i, P_j,\ 0 \leq sent(P_i \to P_j, e_{k,l}) \times (data_{k,l} \times c_{i,j}) \leq 1 \\
\text{(1c)} & \forall P_i, \displaystyle\sum_{P_i \to P_j} \sum_{e_{k,l} \in E_A} \left(sent(P_i \to P_j, e_{k,l}) \times data_{k,l} \times c_{i,j}\right) \leq 1 \\
\text{(1d)} & \forall P_i, \displaystyle\sum_{P_j \to P_i} \sum_{e_{k,l} \in E_A} \left(sent(P_j \to P_i, e_{k,l}) \times data_{k,l} \times c_{j,i}\right) \leq 1 \\
\text{(1e)} & \forall P_i, \displaystyle\sum_{T_k \in V_A} cons(P_i, T_k) \times w_{i,k} \leq 1 \\
\text{(1f)} & \forall P_i, \forall e_{k,l} \in E_A : T_k \to T_l, \\
& \displaystyle\sum_{P_j \to P_i} sent(P_j \to P_i, e_{k,l}) + cons(P_i, T_k) = \\
& \displaystyle\sum_{P_i \to P_j} sent(P_i \to P_j, e_{k,l}) + cons(P_i, T_l)
\end{cases}$$

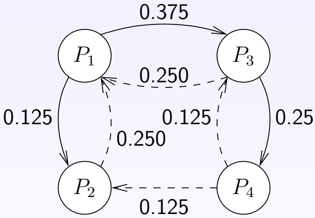
How to design a schedule achieving this throughput?

Back to the example

Computations

	$cons(P_i, T_1)$
P_1	0.025
P_2	0.125
P_3	0.125
P_4	0.250
Total	21 tasks / 40 seconds

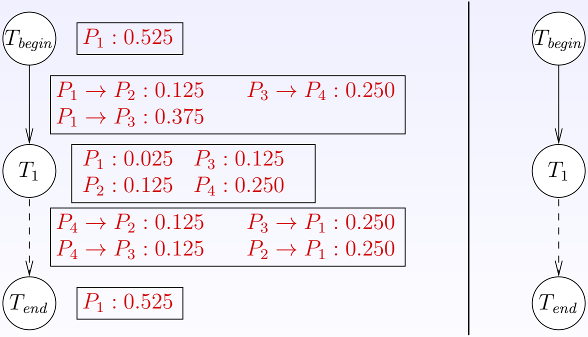
Communications



$$sent(P_i \to P_j, e_{k,l})$$

Decomposition into a set of allocations (1/2)

Steady state = superposition of several allocations



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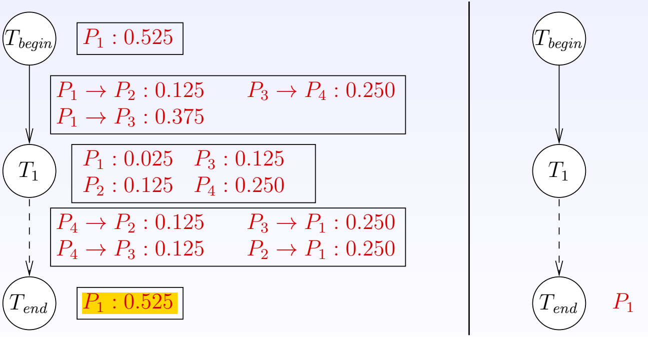

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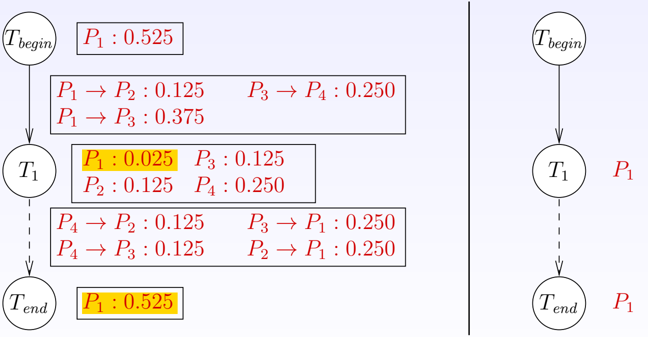

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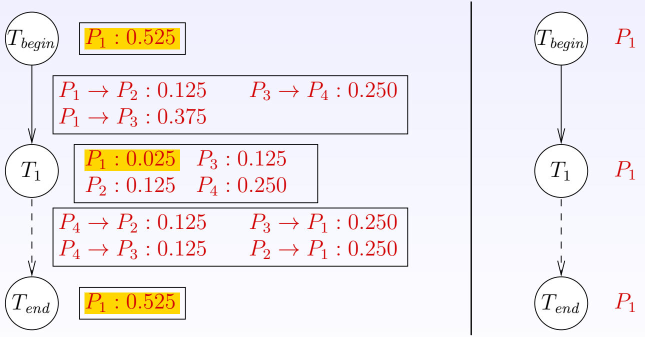

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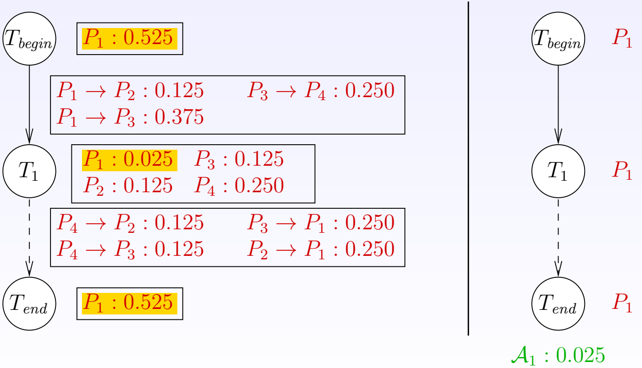

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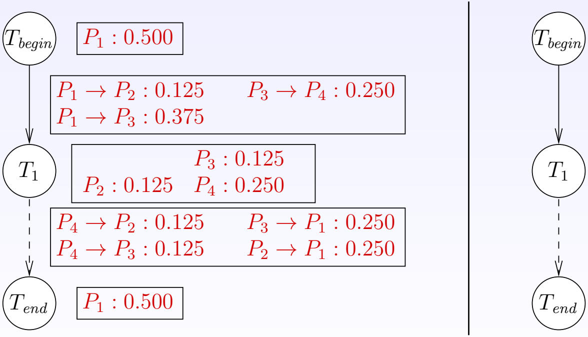

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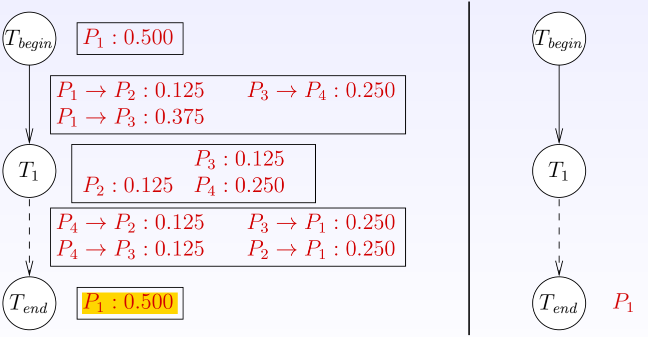

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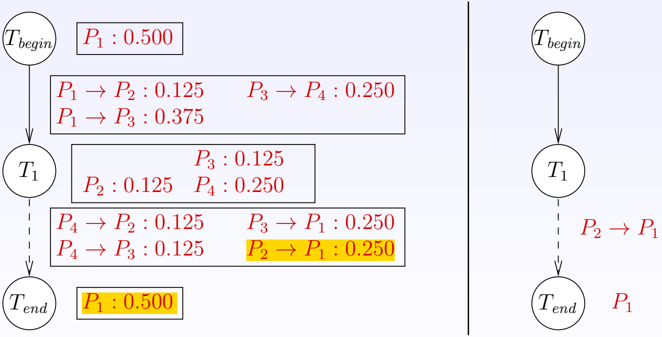

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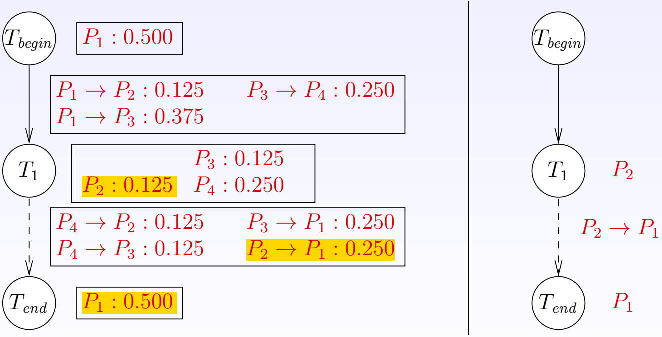

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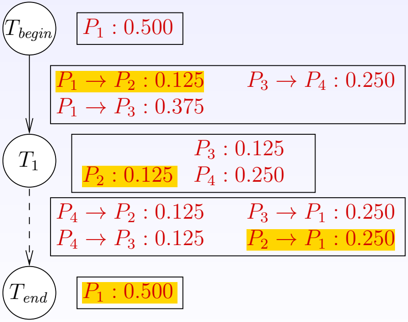

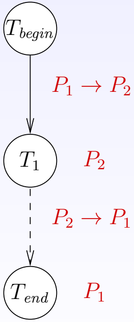

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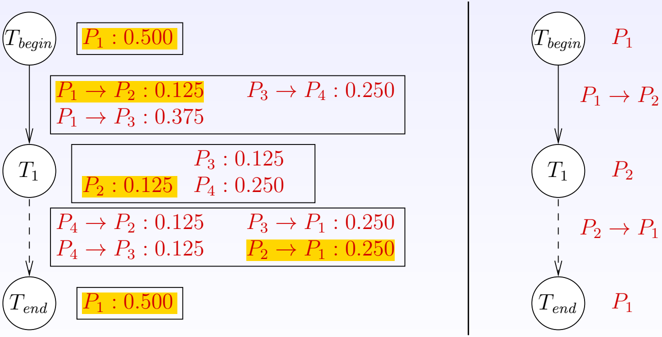
Steady state = superposition of several allocations



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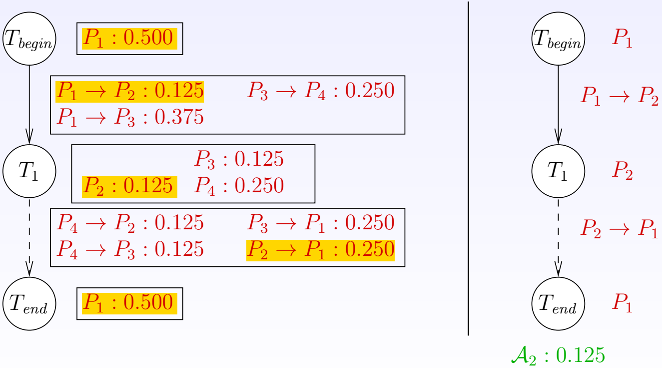

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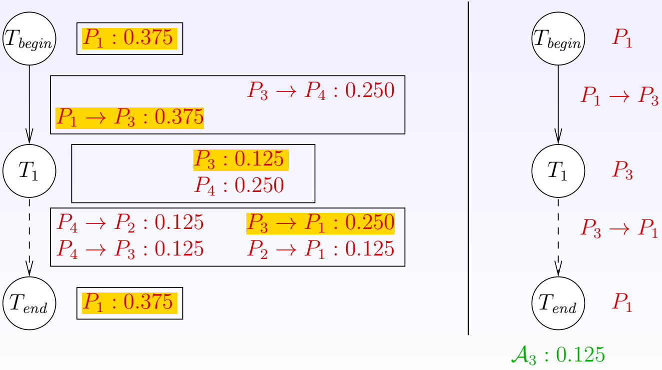

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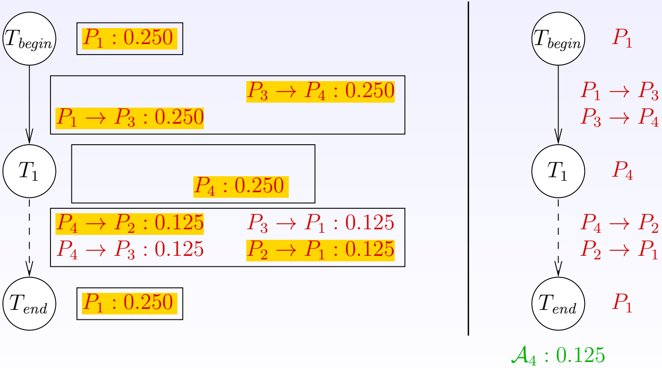

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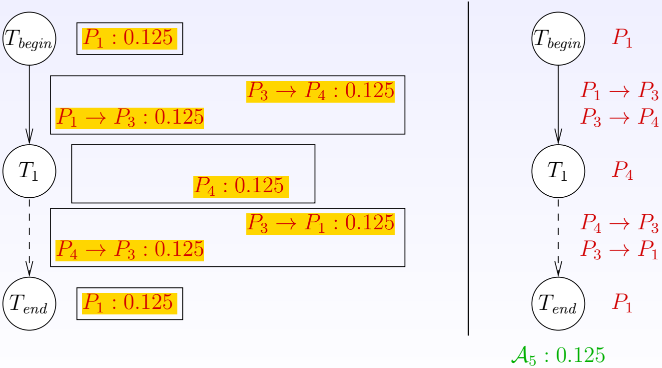
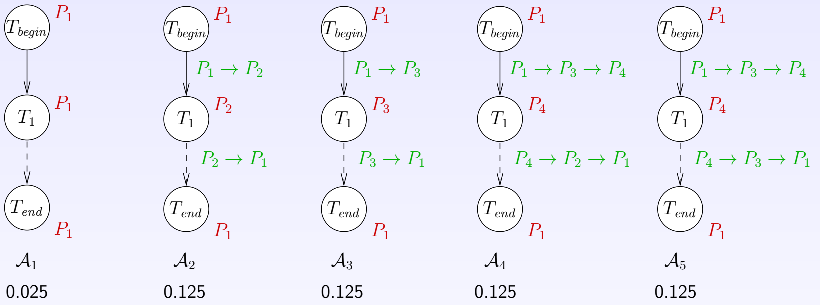
Steady state = superposition of several allocations

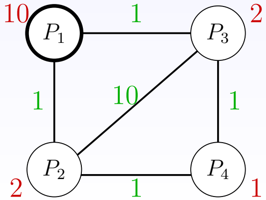


Decomposition into a set of allocations (2/2)

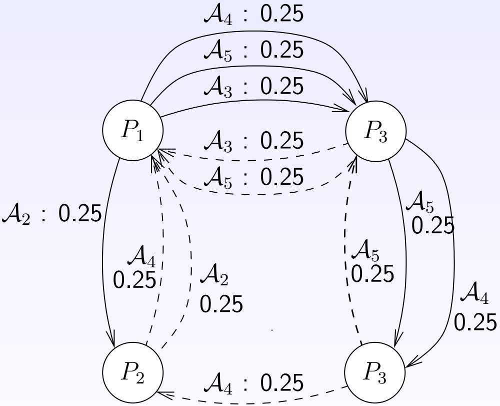


This decomposition is always possible

How to orchestrate these allocations?

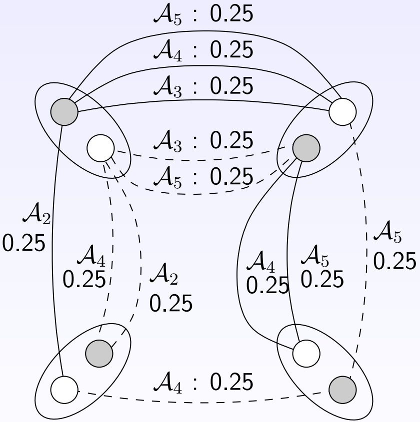


Communication graph

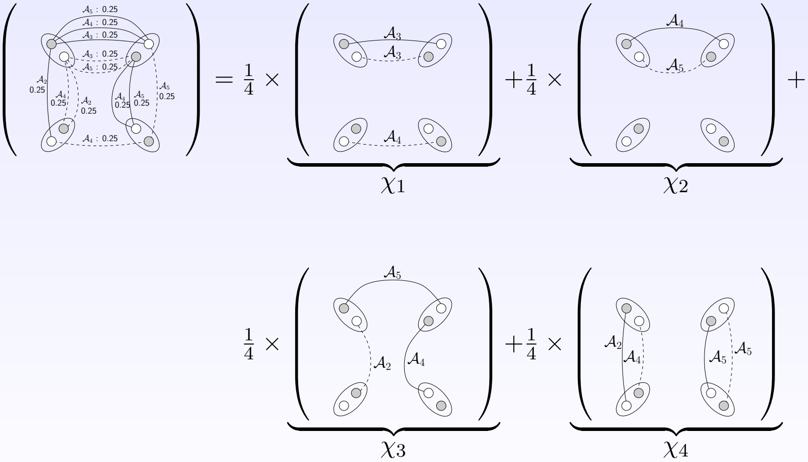


Fraction of time spent transferring some $e_{k,l}$ file from P_i to P_j for a given allocation

One-port constraints = matching

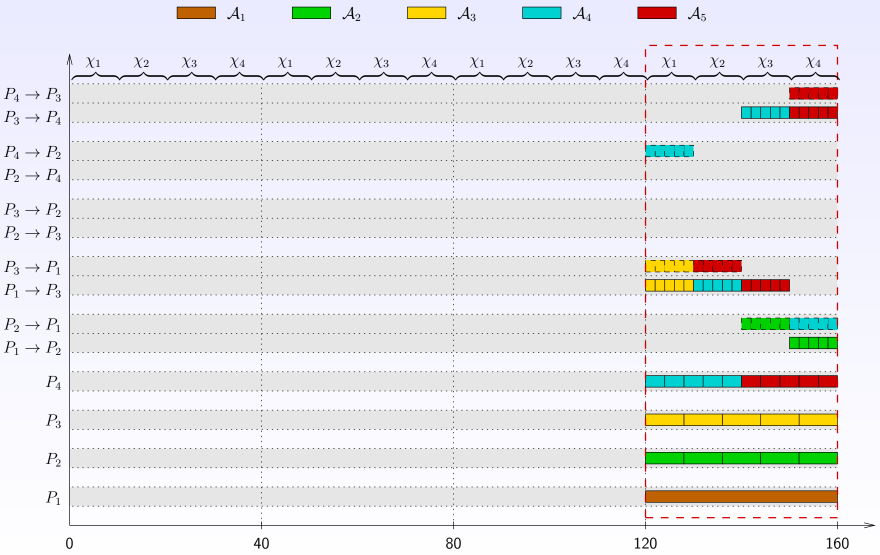


Edge coloring (decomposition into matchings)



This decomposition is always possible

Cyclic scheduling achieving optimal throughput



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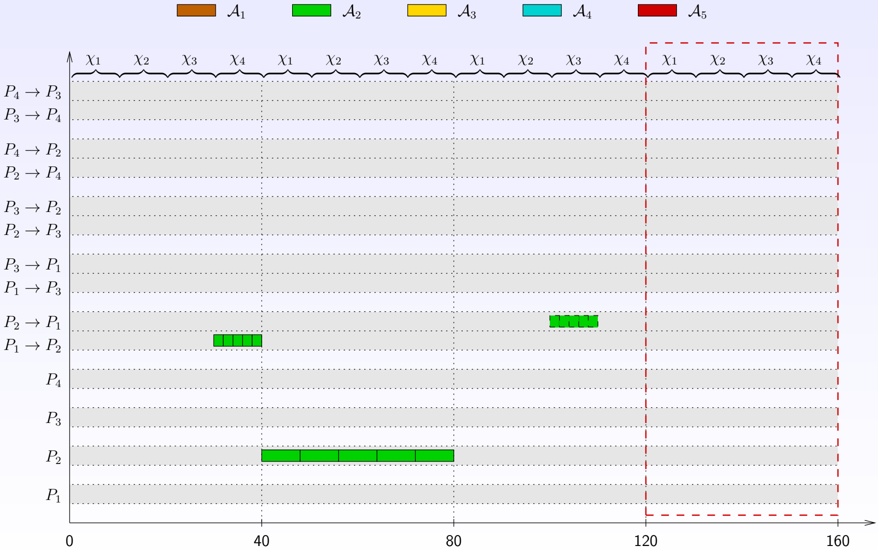

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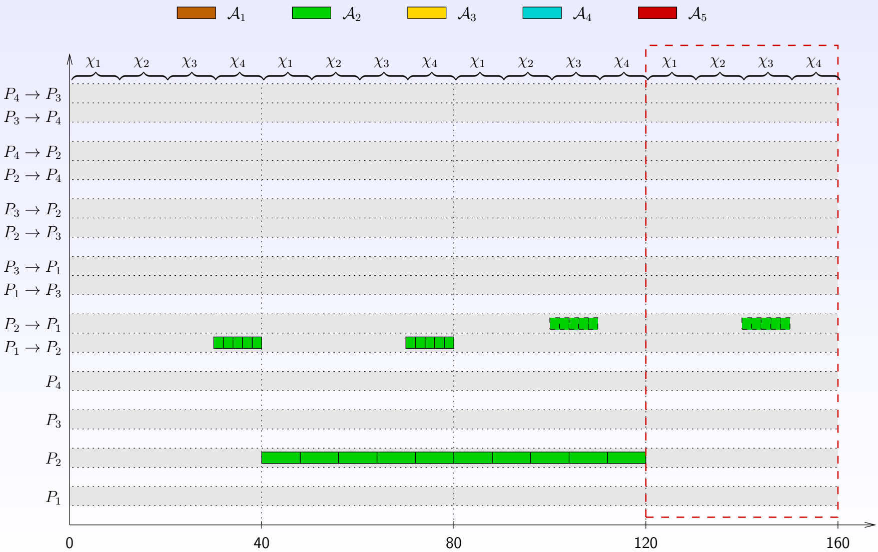

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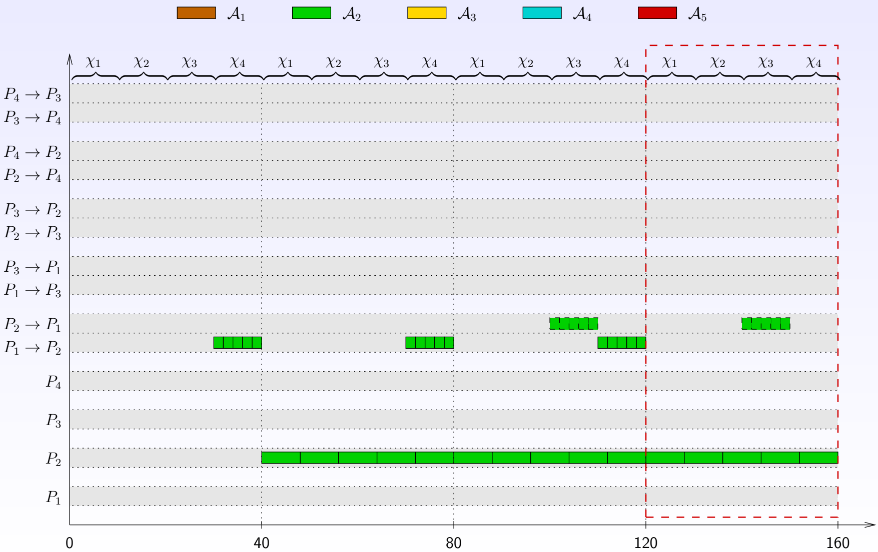

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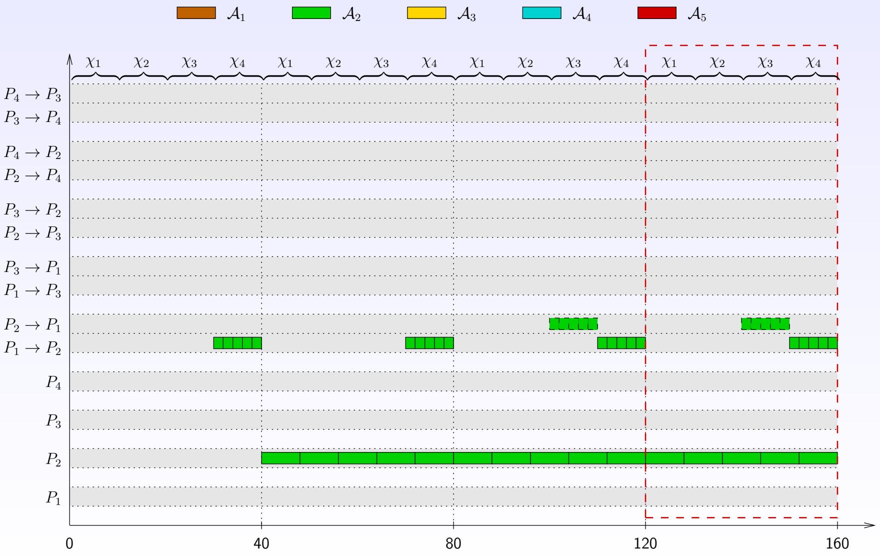

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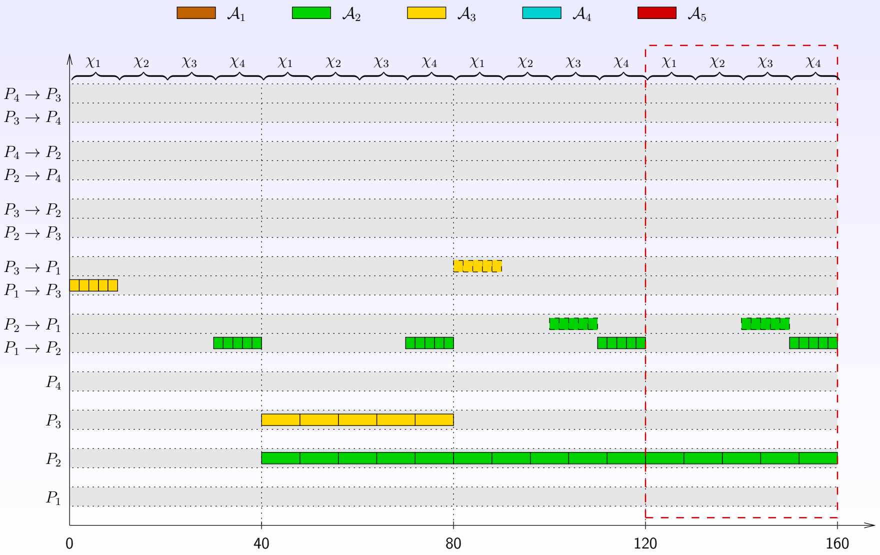

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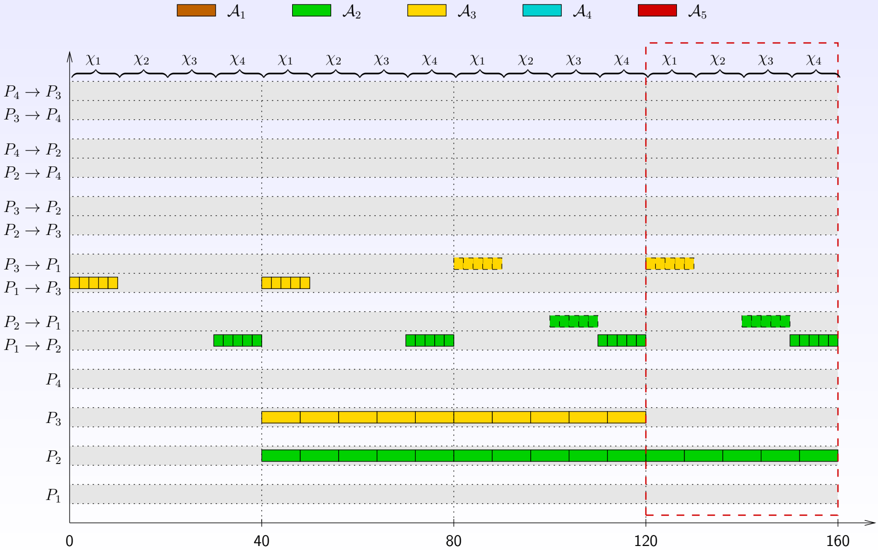

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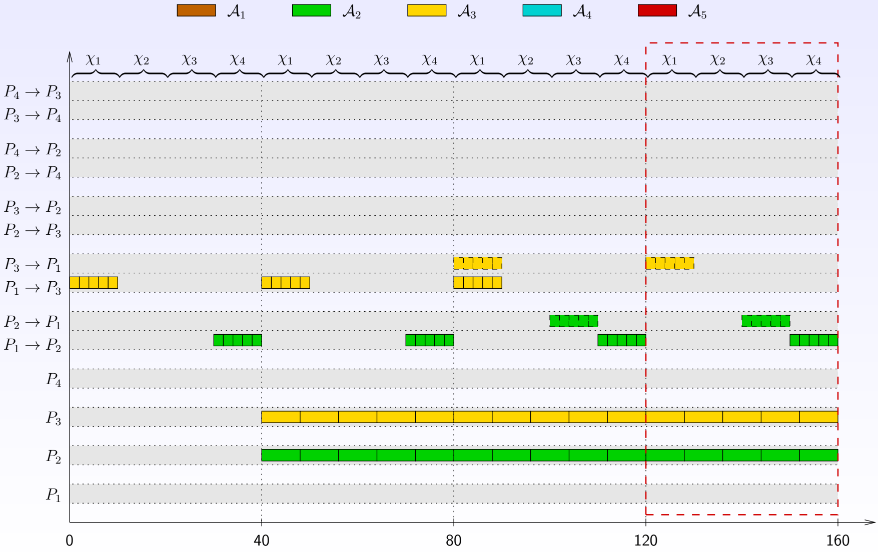

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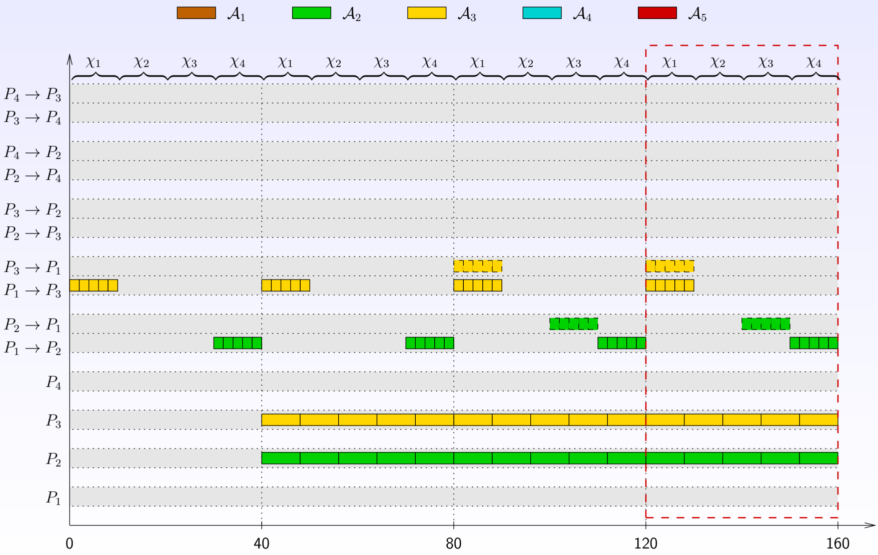

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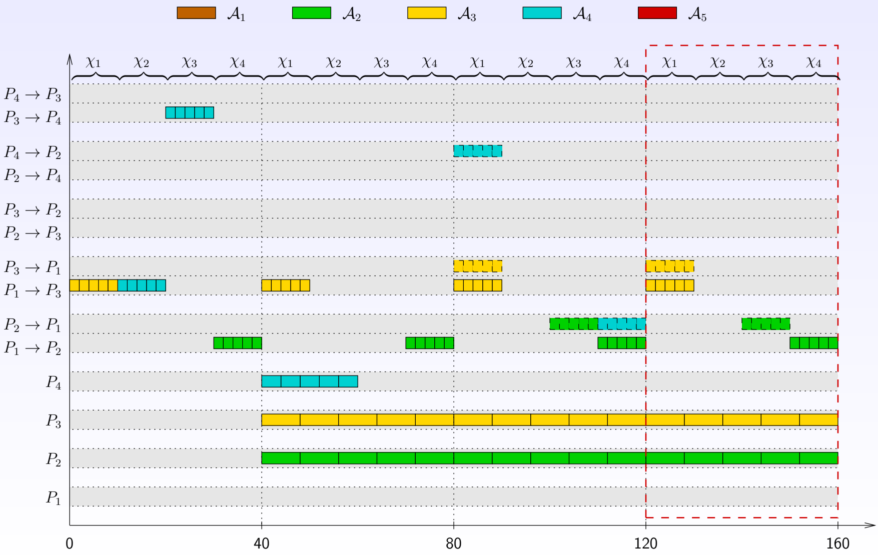

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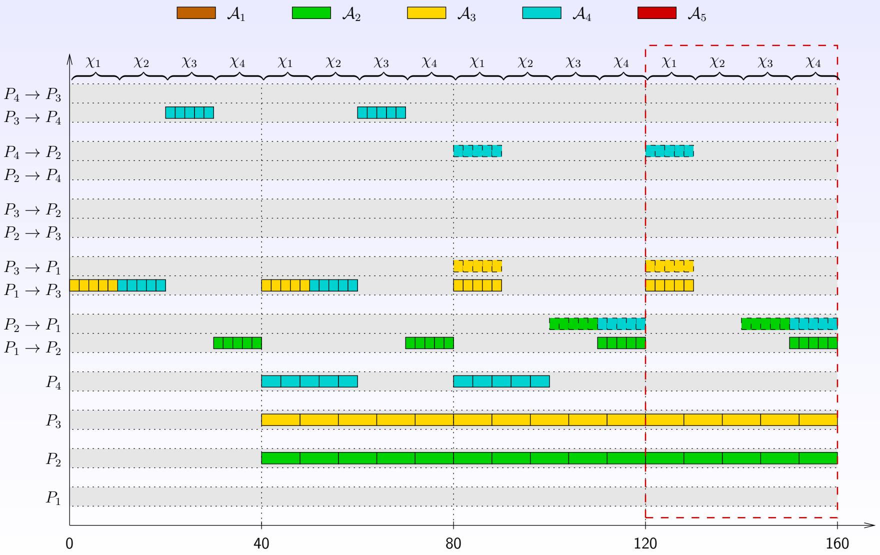

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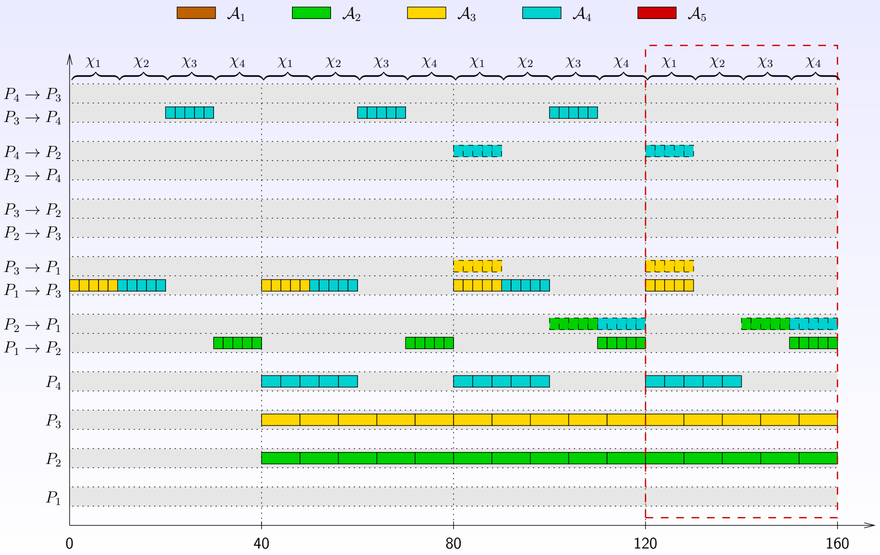

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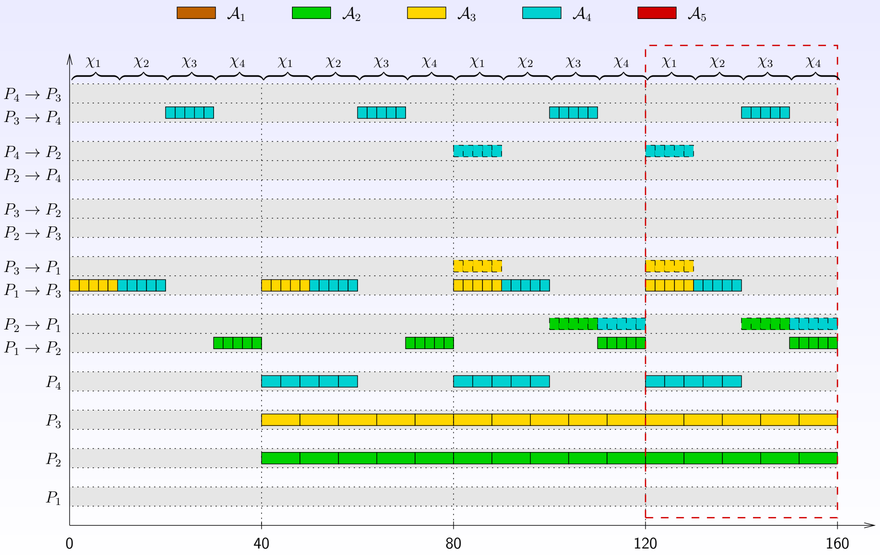

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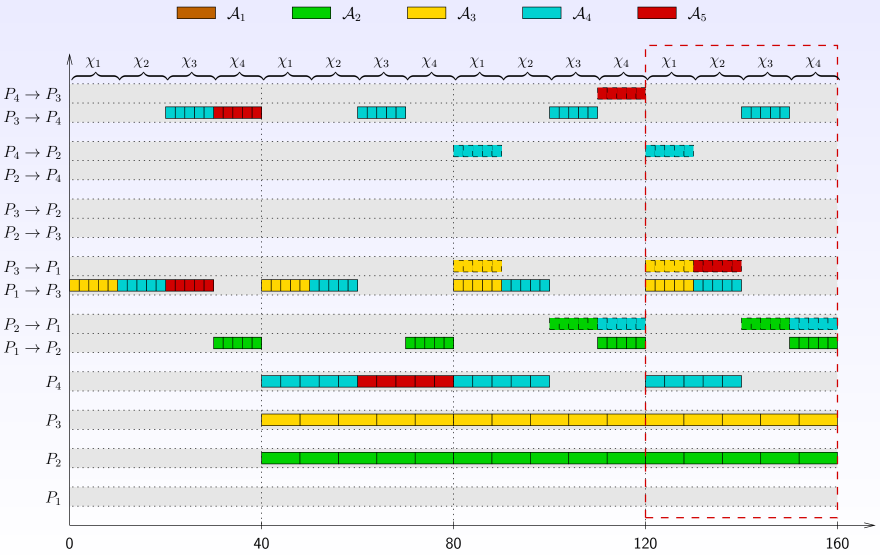

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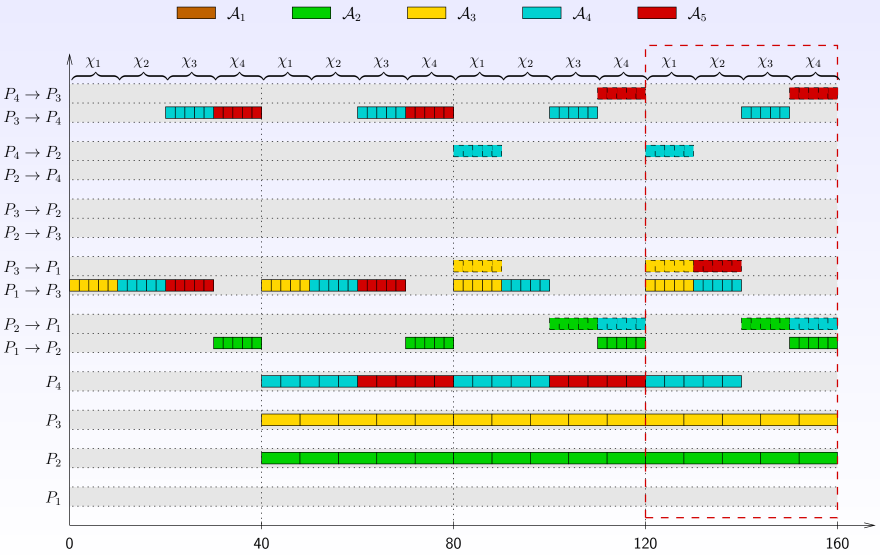

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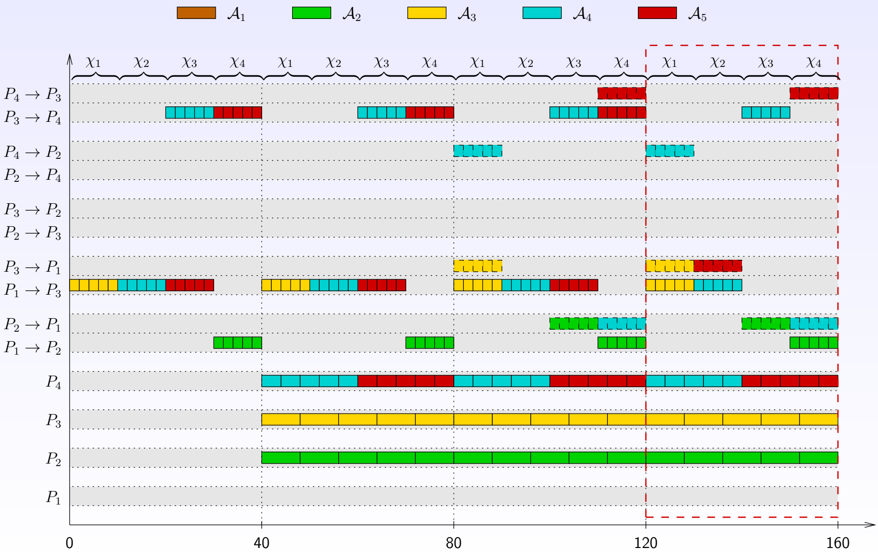
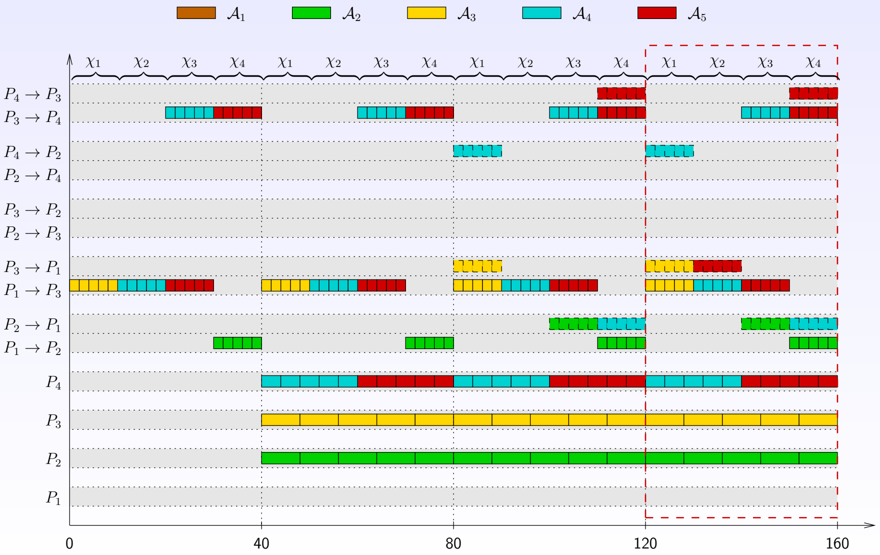
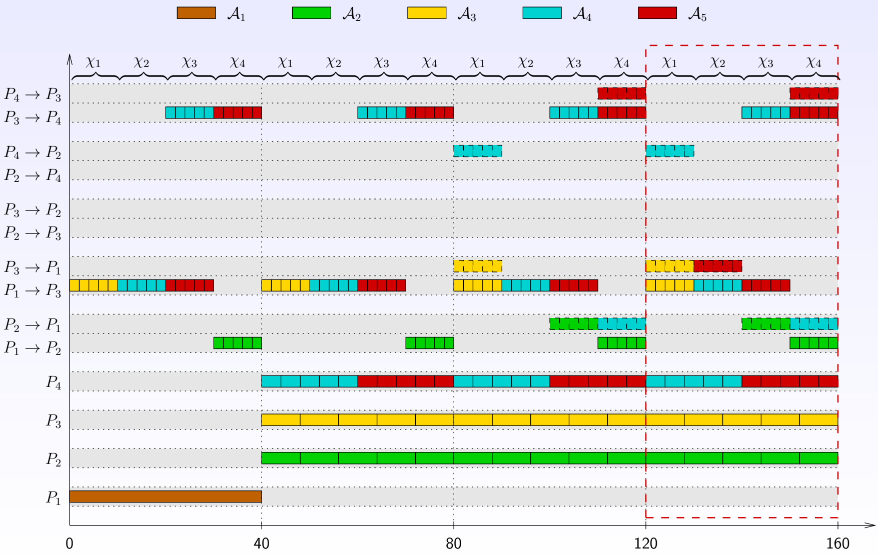


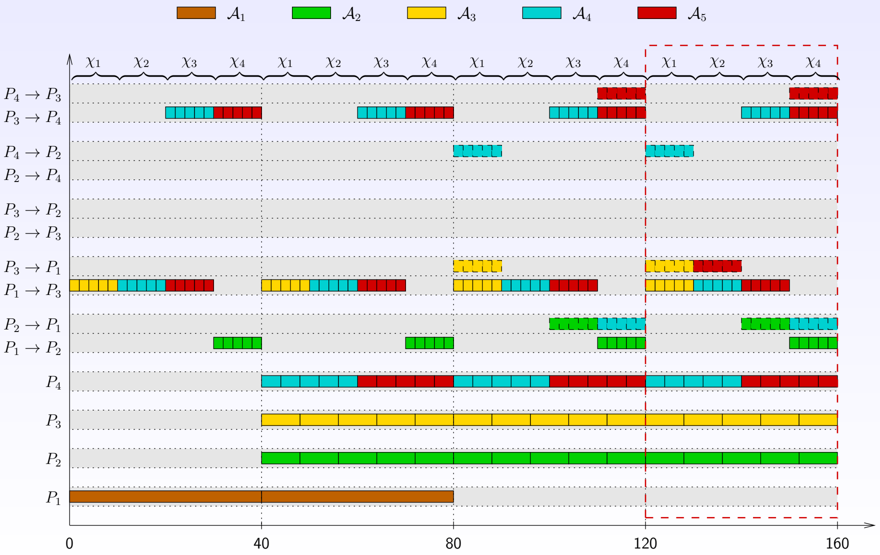
Cyclic scheduling achieving optimal throughput



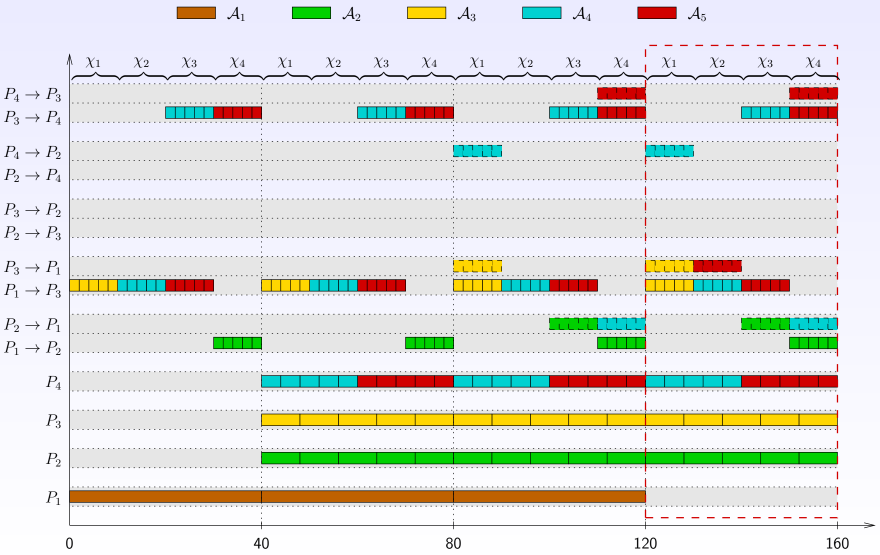
Cyclic scheduling achieving optimal throughput



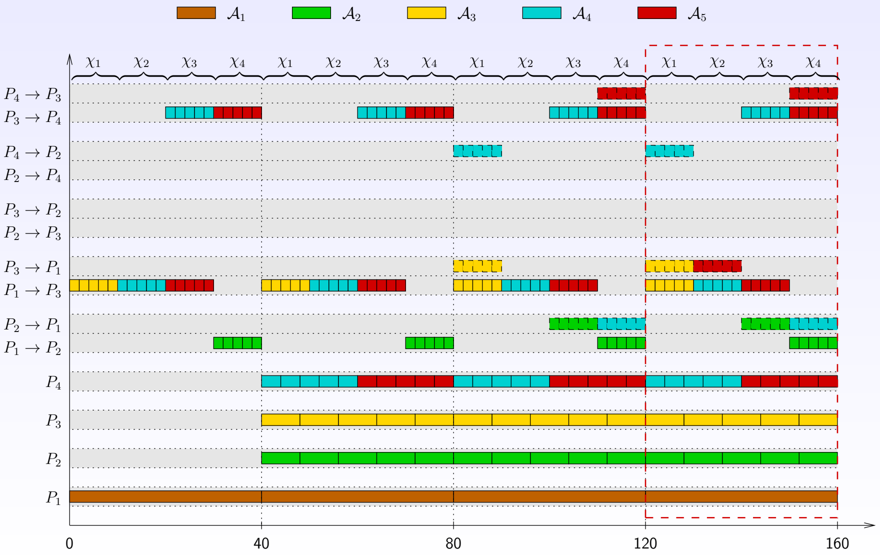
Cyclic scheduling achieving optimal throughput



Cyclic scheduling achieving optimal throughput



Cyclic scheduling achieving optimal throughput



Asymptotically optimal schedule

- The technique used in the example is
  - general
  - polynomial
- The resulting schedule is asymptotically optimal: within T time-steps, it differs from the optimal schedule by a constant number of tasks (independent of T)

Extensions to collections of general task graphs

- More difficult but possible
- Maximizing throughput NP-hard ☹
- Most application DAGs have polynomial number of joins
  $\Rightarrow$ polynomial solution ☺