Power-aware Manhattan routing on chip multiprocessors (rebuttal phase)

Theorem 1: Given a $p \times q$ CMP with $q \geq p$, $q = O(p)$, and a set of communications to be routed from $\mathcal{C}_{1,1}$ to $\mathcal{C}_{p,q}$, the minimum upper bound for the ratio of the power consumed by an XY routing (P_{XY}) over the power consumed by a max-MP routing ($P_{\max}$) is in $O(p)$.

Note that the result holds true for a $p \times p$ square CMP as a particular case. Note also that it holds for the symmetric case of a CMP with $p \geq q$ and $p = O(q)$, with a minimum upper bound in $O(q)$.

Proof: We first prove that an upper bound of $P_{\mathrm{XY}}/P_{\max}$ is in $O(p)$. Then, we show that this bound can indeed be achieved.

Let K be the total size of the communications to route (i.e., $K = \sum_{i \in \{1,\dots,n_c\}} \delta_i$). The XY routing is forwarding all these communications along the same route, leading to a power consumption $P_{\mathrm{XY}} = (p+q) \times K^\alpha$, and therefore P_{XY} is in $O(p \times K^\alpha)$ (recall that $q = O(p)$).

All communications, even if split in multiple paths (as allowed with a max-MP routing), follow the same diagonals in direction 1. For each $k \in \{1, \dots, q+p-2\}$, we let $K_k^{(1)}$ be the sum of the γ_i for all $i \in \{1, \dots, n_c\}$ such that $ksrc(i) \leq k$ and $ksnk(i) > k$. Since all communications have the same source and destination, $K_k^{(1)} = K$ for each k. For a given $K_k^{(1)}$, the ideal way to map those communications is to distribute them among all the communication links from $D_k^{(1)}$ to $D_{k+1}^{(1)}$ (see Figure 3). Such a splitting cannot be achieved but provides a bound on how to load-balance the communication across the links. We have:

$$P_{\max} \geq \sum_{k=1}^{p-1} 2k \left(\frac{K_k^{(1)}}{2k}\right)^\alpha + \sum_{k=p}^{q-1} (2p-1) \left(\frac{K_k^{(1)}}{2p-1}\right)^\alpha + \sum_{k=q}^{q+p-2} 2(q+p-k-1) \left(\frac{K_k^{(1)}}{2(q+p-k-1)}\right)^\alpha,$$

and, since $K_k^{(1)} = K$ and $\sum_{k=1}^{p-1} k^{1-\alpha} \geq \int_1^p dx/x^{1-\alpha}$,

$$P_{\max} \geq K^\alpha \left(2 \times \frac{1}{2^{\alpha-1}} \frac{1}{2-\alpha} \left(1 - p^{2-\alpha}\right) + \frac{q-p}{(2p-1)^{\alpha-1}}\right),$$

and hence $P_{\max} = O(K^\alpha)$, since $\alpha > 2$ and $q = O(p)$.

Finally, since $P_{\mathrm{XY}} = O(p \times K^\alpha)$, we conclude that the worst ratio $P_{\mathrm{XY}}/P_{\max}$ is at most in $O(p)$, hence providing an upper bound on this ratio.

We now exhibit an instance of the problem on a $p \times q$ CMP, such that $q = O(p)$ and $q \geq p$, and a max-MP routing such that the ratio $O(p)$ is realized, when all communications go from the same source core $\mathcal{C}_{1,1}$ to the same destination core $\mathcal{C}_{p,q}$. Let $p = 2 \times p'$, and K be the total size of the communications to route. The power consumed with an XY routing is $P_{\mathrm{XY}} = (p+q) \times K^\alpha$.

Now we consider the max-MP routing pattern based on Figure 4. Until semi-diagonal $D_{2p'}^{(1)}$, communications are split according to the figure. Then the communications that arrive (there are p' of them) at $D_{2p'}^{(1)}$ are forwarded horizontally. When they reach $D_q^{(1)}$, communications are aggregated according to the symmetrical pattern of the figure.

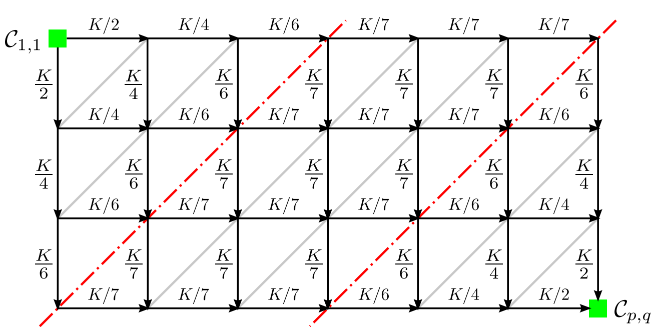

Figure 3. Ideal sharing of one communication.

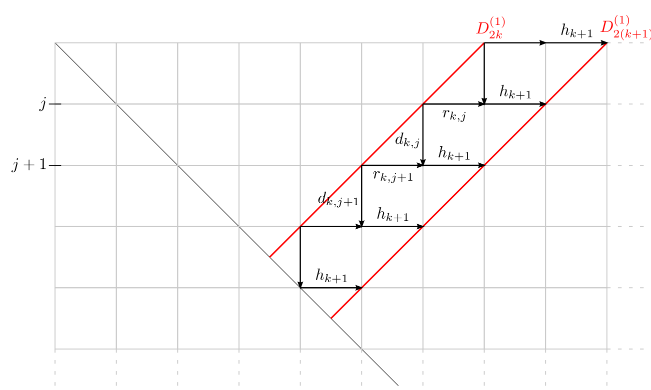

Figure 4. Routing pattern.

We first compute $P_{\max}^{(1)}$, the dissipated power at both ends, where the communications are not forwarded horizontally. We deal with the cores in diagonal. On semi-diagonal $D_{2k}^{(1)}$, for $j \in \{1, \dots, k\}$, the core $\mathcal{C}_{j,2k+1-j}$ on line j is sending $r_{k,j}$ communications to its right core, and $d_{k,j}$ to its down core. Between $D_{2k}^{(1)}$ and $D_{2(k+1)}^{(1)}$, for $j \in \{1, \dots, k+1\}$, the core $\mathcal{C}_{j,2k+2-j}$ on line j is sending h_{k+1} communications to its right core.

We set:

- for $k \in \{1, \dots, p'\}$, $h_k = \dfrac{K}{k}$;
- for $k \in \{1, \dots, p'-1\}$ and $j \in \{1, \dots, k\}$,

$$r_{k,j} = \frac{k+1-j}{k(k+1)} K \quad \text{and} \quad d_{k,j} = \frac{j}{k(k+1)} K \ .$$

We show that the splits and merges of communications are valid:

- for $k \in \{1, \dots, p'-1\}$ and $j \in \{2, \dots, k\}$,

$$\frac{1}{K}\left(r_{k,j} + d_{k,j-1}\right) = \frac{k}{k(k+1)} = h_{k+1} \ ;$$

- for $k \in \{1, \dots, p'-1\}$, $r_{k,1} = h_{k+1}$ and $d_{k,k} = h_{k+1}$;
- for $k \in \{1, \dots, p'-1\}$ and $j \in \{1, \dots, k\}$,

$$\frac{1}{K}\left(r_{k,j} + d_{k,j}\right) = \frac{k+1}{k(k+1)} = h_k \ .$$

What is the dissipated power with this max-MP routing? The power consumption $P_{\max}^{(1)}$ is twice the power consumed until diagonal $D_{2p'}^{(1)}$ (we define symmetrical routes for the other half of the routing). Therefore, we have:

$$\begin{aligned}
\frac{1}{2} P_{\max}^{(1)} &= \sum_{k=1}^{p'} k\left(h_k\right)^\alpha + \sum_{k=1}^{p'-1} \sum_{j=1}^{k} \left(\left(d_{k,j}\right)^\alpha + \left(r_{k,j}\right)^\alpha\right) \\
&\leq \sum_{k=1}^{p'} k\left(h_k\right)^\alpha + \sum_{k=1}^{p'-1} \sum_{j=1}^{k} \left(d_{k,j} + r_{k,j}\right)^\alpha \ .
\end{aligned}$$

Also, we know that for $k \in \{1, \dots, p'-1\}$ and $j \in \{1, \dots, k\}$, $d_{k,j} + r_{k,j} = h_k$. Therefore,

$$\begin{aligned}
\frac{1}{2} P_{\max}^{(1)} &\leq \sum_{k=1}^{p'} k\left(h_k\right)^\alpha + \sum_{k=1}^{p'-1} k\left(h_k\right)^\alpha \leq 2K^\alpha \sum_{k=1}^{p'} \frac{1}{k^{\alpha-1}} \\
&\leq 2K^\alpha \left(1 + \left(1 - 1/p'\right)\right) \ .
\end{aligned}$$

Now, the power dissipated in the horizontal links in the middle of the CMP is:

$$P_{\max}^{(2)} = (q-p)p' \times (K/p')^\alpha \leq K^\alpha \times q(p')^{1-\alpha}.$$

There are indeed p' communications of size K/p', each of length $(q-p)$. Altogether,

$$P_{\max} = P_{\max}^{(1)} + P_{\max}^{(2)} \leq K^\alpha \left(8 - 4/p' + q(p')^{1-\alpha}\right).$$

Since $q = O(p')$ and $\alpha > 2$, we have $8 - 4/p' + q(p')^{1-\alpha} = O(1)$, and since $P_{\text{XY}} = (p+q) \times K^\alpha$ and $q = O(p)$, the ratio $P_{\text{XY}}/P_{\max}$ is in $O(p)$, which concludes the proof. ∎